Exact Results for 't Hooft Loops on S^4 and $S^1 \times \mathbb{R}^3$

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Plan

SUSY gauge theory.

- $\mathcal{N} = 1, 2, \ldots$
- Known exact results
- Non-local observables
- Relations to 2D theories

- 't Hooft loops on S^4

(- Localization in $S^1 \times \mathbb{R}^3$ and wall crossing)

- Conclusions

Based on

arXiv: 1105.2568 with J Gomis & V. Pestun

Work in progress w/ Y. Ito & M. Taki

SUSY gauge thy

$Q(\text{boson}) = \text{fermion}$

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$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = P^\mu \sigma_{\mu\alpha\dot\beta}$$

$$\{Q_\alpha, Q_\beta\} = 0$$

$N = 1$

$\Longrightarrow$

$$\{Q_\alpha^{I=1,2,\ldots,N}, \bar{Q}_{\dot\beta, J}\} = \delta^I_J \, P^\mu \sigma_{\mu\alpha\dot\beta}$$

$$\{Q_\alpha^I, Q_\beta^J\} = Z^{[IJ]} \epsilon_{\alpha\beta}$$

$\mathcal{N}=2$ vector multiplet

$$A_\mu$$

$$\lambda \qquad \psi \qquad\qquad \leftarrow SU(2)_R$$

$$\phi$$

$\mathcal{N}=2$ hypermultiplet

$$Q$$

$$\psi_q \qquad \psi_{\tilde{q}}^\dagger \qquad\qquad \leftarrow SU(2)_R$$

$$\tilde{Q}^\dagger$$

· $\mathcal{N}=1$ vector & chiral multiplets

Non-chiral $\Rightarrow$ ~~SR~~ sector

Known exact results

· Witten '88.

Topological twist of $\mathcal{N}=2$ gauge theory.

$$SU(2)_{left} \times [SU(2)_{right} \times SU(2)_R]_{diag}$$

as Lorentz group

Scalar
$\rightsquigarrow$ Supercharge

Curved manifold M_4

$$S = \frac{1}{g^2} Q(\ldots) + \# \cdot \tau \int_{M_4} \mathrm{Tr}\, F \wedge F$$

Donaldson inv.

Localization

$$= \int \mathcal{D}A \, e^{-S} \ldots = \sum_k q^k \int_{M_{inst,k}(M_4)} (\ldots)$$

$\Leftarrow$ No IR divergence

'94 Seiberg & Witten

Exact low-energy effective action for $\mathbb{R}^4$

$$S = \frac{1}{4\pi}\int d^4x \, \mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^2} W_\alpha W^\alpha\right]$$

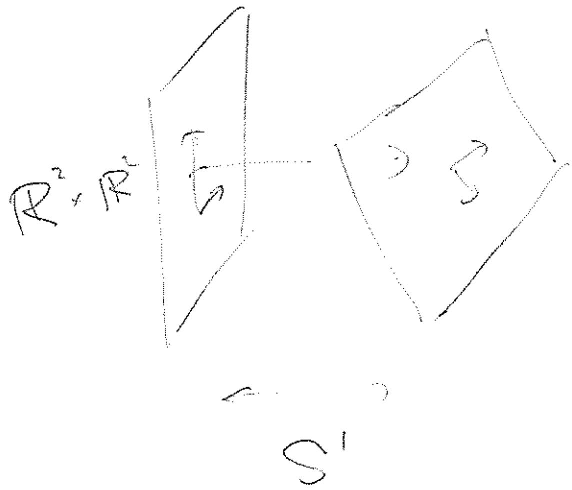
$\mathcal{F}$: pre-potential

- Consistency from holomorphy
- Direct path-integral derivation?

$\exists$ IR divergence.

'02 Nekrasov

IR regulator : Ω- deformation



Rotation parameters ϵ_1 & ϵ_2.

Dim red of 5D thy on S^1.

$$Z_{inst}(\epsilon_1, \epsilon_2, a) = \int \mathcal{D}A \; e^{-S}$$

$$= \sum_{k \geq 0} q^k Z_k(\epsilon_1, \epsilon_2, a)$$

As $\epsilon_1, \epsilon_2 \to 0$ $\quad Z_{inst} \sim e^{\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{inst}(a)}$

Direct determination of the pre potential.

- In the last few years, new kinds of exact results in SUSY gauge theories started to appear.

SUSY gauge theories on curved manifolds.

c.f. Seiberg's talk

Nagasaki & Yamaguchi

Prototype :

Pestun's localization on S^4

Physical $\mathcal{N}=4$ SYM on $\mathbb{R}^4$ conformal

$\rightsquigarrow$ write physical action on S^4.

$\exists$ SUSY. $PSU(2,2|4)$

Introduce mass preserving $OSp(2|4)$

$\mathcal{N}=2$ on S^4.

Pestun computed

$$\int \mathcal{D}A \ldots \; \mathrm{Tr}_R\, e^{-\oint (A+i\Phi)}\; e^{-S - tQ\cdot V}$$

$$t \to \infty$$

$$V = \langle \Psi, \overline{Q\Psi} \rangle$$

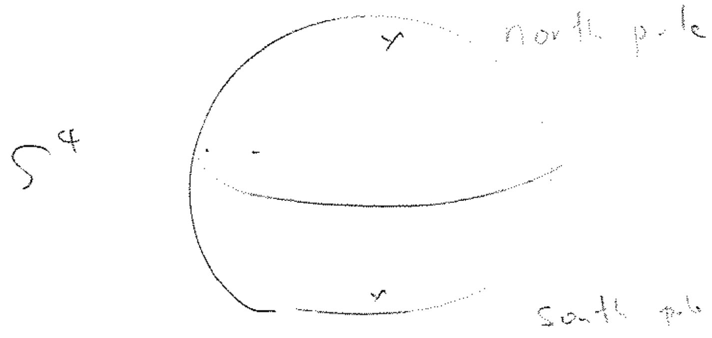
Without mass this reduces to

$$\int dM\; e^{-\frac{1}{g}\mathrm{Tr}\,M^2}\, \mathrm{Tr}_R\, e^{M}$$

Erickson Semenoff Zarembo
Drukker Gross

· Case with non-zero mass. Very complicated

$$Z_{S^4} = \int da\; Z_{1\text{-loop}}(a, m)\, \left| Z_{inst}(a, \tau, m) \right|^2 e^{-S_{cl}(a, \ldots)}$$



Connection to 2D theory

Alday, Gaiotto & Tachikawa

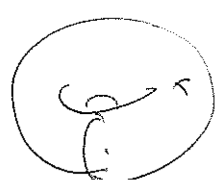
$Z_{1\text{-loop}}(a,m)$ = "DOZZ 3-pt fun of Liouville thy"

$G=SU(2)$

$Z_{inst}(a,m,\tau)$ = Conformal block of Liouville thy

Z_{S^4} = Correlation function of Liouville thy.

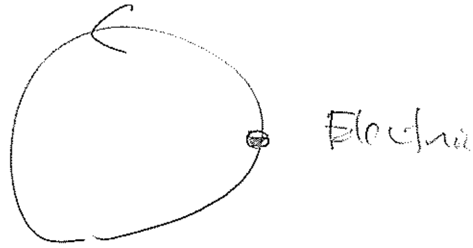
$\langle \mathrm{Tr}_R \, e^{-\oint(A+i\Phi)} \rangle = \langle$ Topological defect $\rangle_{\text{Liouville}}$



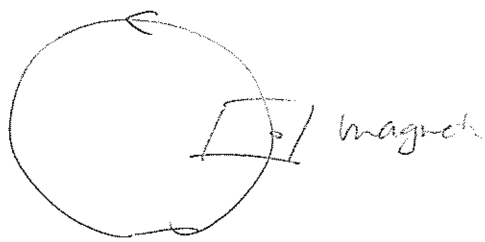
Wilson loop

$$W = \mathrm{Tr}\, e^{-\oint A \,(+ i \,\Phi\, ds)}$$

Confinement



't Hooft loop



Higgs mechanism

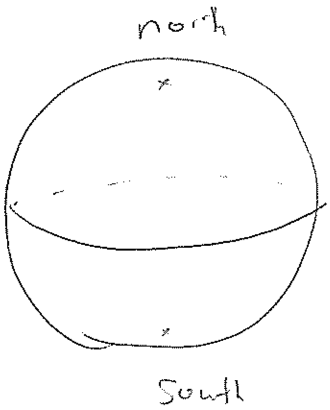
$$\langle T_B \rangle = \ldots$$

$$F = \frac{B}{2}\,\mathrm{vol}(S^2)$$

$$\Phi = \frac{B}{2\pi}$$

- Electromagnetic (S) duality
- Modular transformation

Localization for 't Hooft loops on S^4
(Gomis, T.O., Pestun)



$S^1 \subset$ Equator

In certain coords

$$F = -\frac{B}{2}\,\mathrm{vol}(S^2)$$

$$\Phi_q = \frac{B}{2|\vec{x}|}$$

Solves $Q\Psi = 0$.

$$\langle T_B \rangle = \int_{BC} \mathcal{D}A\, \mathcal{D}\Psi\, e^{-S - tQ\cdot V}$$

$$V = \langle \Psi, \overline{Q\Psi} \rangle$$

$$= \int da \left| \left(e^{-S_{cl}^{pole}}\, Z^{pole}_{1\text{-loop}}\, Z_{inst} \right)\Big|_{a \to a + i\frac{B}{2}} \right|^2 \times Z^{eq}_{1\text{-loop}}$$

$+$ non-pert corr.

monopole screening

Kronheimer

U(1)-inv instantons

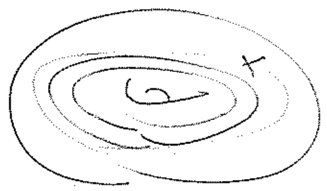
= Singular monopoles

SU(2) N=2*

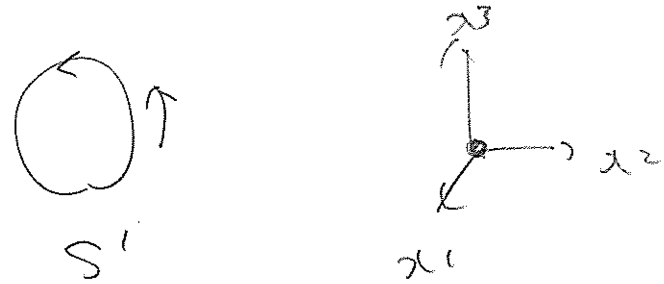
$$\langle T_p \rangle = \sum_{q=-p,-p+2,\dots,p} \frac{p!}{\left(\frac{p+q}{2}\right)!\left(\frac{p-q}{2}\right)!} \int da \left| Z_{cl+1\text{-}loop+inst}\left(a+i\frac{q}{4}\right)\right|^2$$

$$\times \frac{\sinh^{\frac{p}{2}}\left[\pi\left(2a+m+\frac{i}{2}p\right)\right]\sinh^{\frac{p}{2}}\left[\pi\left(2a-m+\frac{i}{2}p\right)\right]}{\sinh^{p}\left[2\pi a+\frac{i}{2}\pi p\right]}$$

· Agrees with Liouville result for



Loop operators on $S^1 \times \mathbb{R}^3$



Neighborhood of loop in $\tilde{S}^4$

Captures $Z^{eq}_{1\text{-loop}}$ and non-pert effects

$\rightsquigarrow$ Monopole analog of the Nekrasov function

- $S^1 \times \mathbb{R}^3$ and line operators were studied by Gaiotto, Moore and Neitzke.
- Framed BPS states :
    - Line op preserves four out of eight supercharges.
    - Modified SUSY algebra. $\exists$ BPS condition. $E \geq -\mathrm{Re}(\zeta^{-1} Z)$

$$\langle L \rangle = \mathrm{Tr}_{\mathcal{H}_L} (-)^F (-y)^{2J_3 + 2J_3} \cdots$$

counts framed BPS states

Can be expanded in IR line ops. Wall-crossing

For $y=1$, they found that $\langle L \rangle$ is given by a holonomy of a (twisted) Hitchin system

$$\langle W \rangle = \sqrt{ZY} + \sqrt{\frac{Z}{Y}} + \frac{1}{\sqrt{ZY}}$$

$$\langle T \rangle = \sqrt{XZ} + \sqrt{\frac{X}{Z}} + \frac{1}{\sqrt{XZ}} \qquad XYZ = -e^{2\pi i m}$$

$$\langle WT \rangle = \sqrt{XY} + \sqrt{\frac{Y}{X}} + \frac{1}{\sqrt{XY}}$$

Satisfy $\langle W \rangle^2 + \langle T \rangle^2 + \langle WT \rangle^2 - \langle W \rangle \langle T \rangle \langle WT \rangle = 4\cos^2 \pi m$

Our results

$$\langle W \rangle = e^{2\pi i a} + e^{-2\pi i a}$$

$$\langle T \rangle = \left(e^{2\pi i b} + e^{-2\pi i b}\right) \frac{\sin^{\frac{1}{2}}(2\pi a + \pi m)\,\sin^{\frac{1}{2}}(2\pi a - \pi m)}{\sin 2\pi a}$$

$$\langle WT \rangle = \left(b \to b + a \right)$$

Satisfy the same relation

In fact, a and b are complexified Fenchel-Nielsen coordinates

$y \neq 1 \Rightarrow$ Non-commutative algebra

Conclusions

- $\mathcal{N}=2$ SUSY gauge thy is a rich subject for mathematical physics

- Non-local ops are useful to study duality and non-perturbative physics
- Developed methods to perform path integral calc for 't Hooft loop.
- Found non-trivial agreement with 2D theory predictions