Atiyah-Patodi-Singer index from the domain-wall Dirac operator

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based on
Hidenori Fukaya, Tetsuya Onogi, SY,
Phys. Rev. D 96, 125004 (2017), arXiv:1710.03379

Introduction

Index theorem of Dirac operators (Atiyah-Singer)

A theorem on the number of solutions

$$D\psi = 0 \qquad D := \gamma^\mu(\partial_\mu + iA_\mu)$$

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

Index theorem appears in various situations in physics

- Number of generations in compactification.

- Anomaly. $\psi' = e^{i\alpha\gamma_5}\psi$

$$\int D\psi' D\bar{\psi}' = \int D\psi D\bar{\psi} e^{i\alpha \mathrm{Ind}(D)}$$

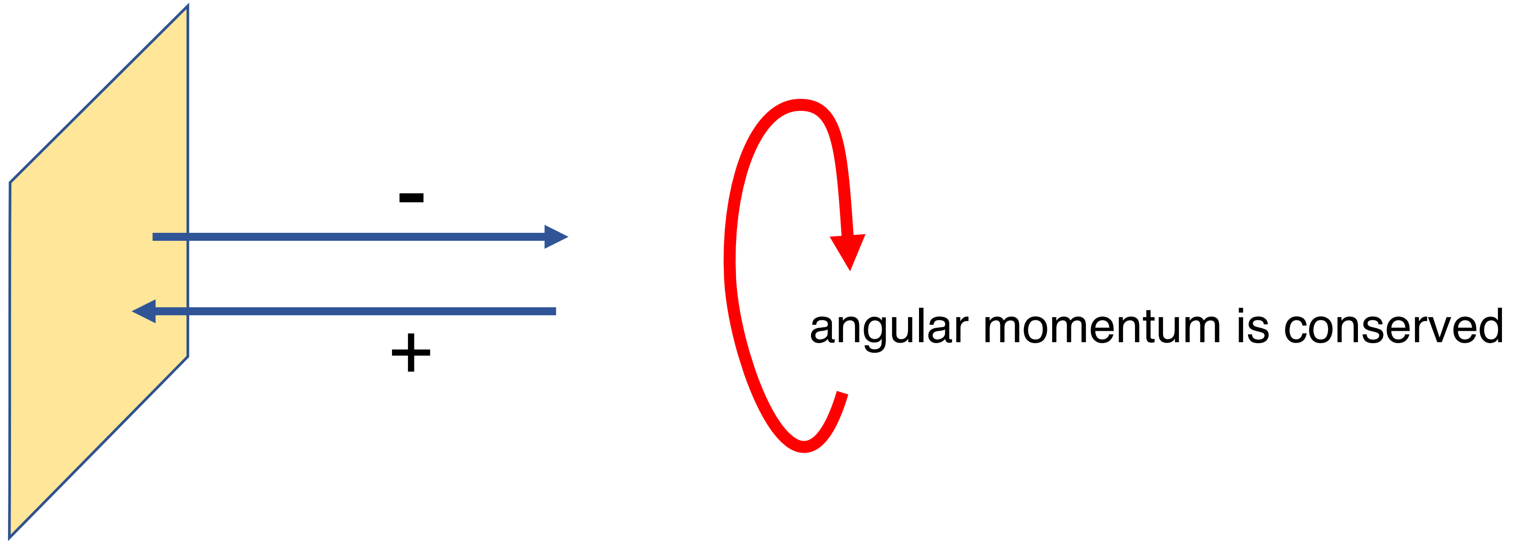
- Topological material (Discussed later).

Any index theorem with **boundary?**

- Number of generations in compactification with **boundary?**
- Anomaly with **boundary?**
- Topological material with **boundary?**

Difficulty of index with boundary

If we impose **local** and **Lorents (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer (APS) boundary condition

[Atiyah, Patodi, Singer 75]

Abandon the locality and preserve the chirality.

Eg. 4 dim $\quad x^4 \geq 0$

$A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose

$$(A + |A|)\psi|_{x^4=0} = 0$$

APS index theorem [Atiyah, Patodi, Singer 75]

$$\mathrm{Ind}(D) = \frac{\eta(iD^{3D})}{2} + \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

“eta invariant” (discussed later)

Any application to physics?

cf. [Alvarez-Gaume, Della Pietra, Moore 85]

Topological insulator.

- 4 dim massive fermion.
- CP symmetry is imposed.

Two distinct "phases" according to the sign of mass.

$$S = \int d^4x \bar{\psi}(D \pm M)\psi$$

Non-trivial phase with boundary has massless edge modes.

Topological insulator and APS index

[Witten 15]

$$S = \int d^4x \bar{\psi}(D \pm M)\psi$$

Partition function

$$Z = \int D\psi D\bar{\psi} e^{-S}$$

APS index

Trivial phase	Topological insulator
$Z = \|Z\|$	$Z = \|Z\|(-1)^{\mathrm{Ind}(D)}$

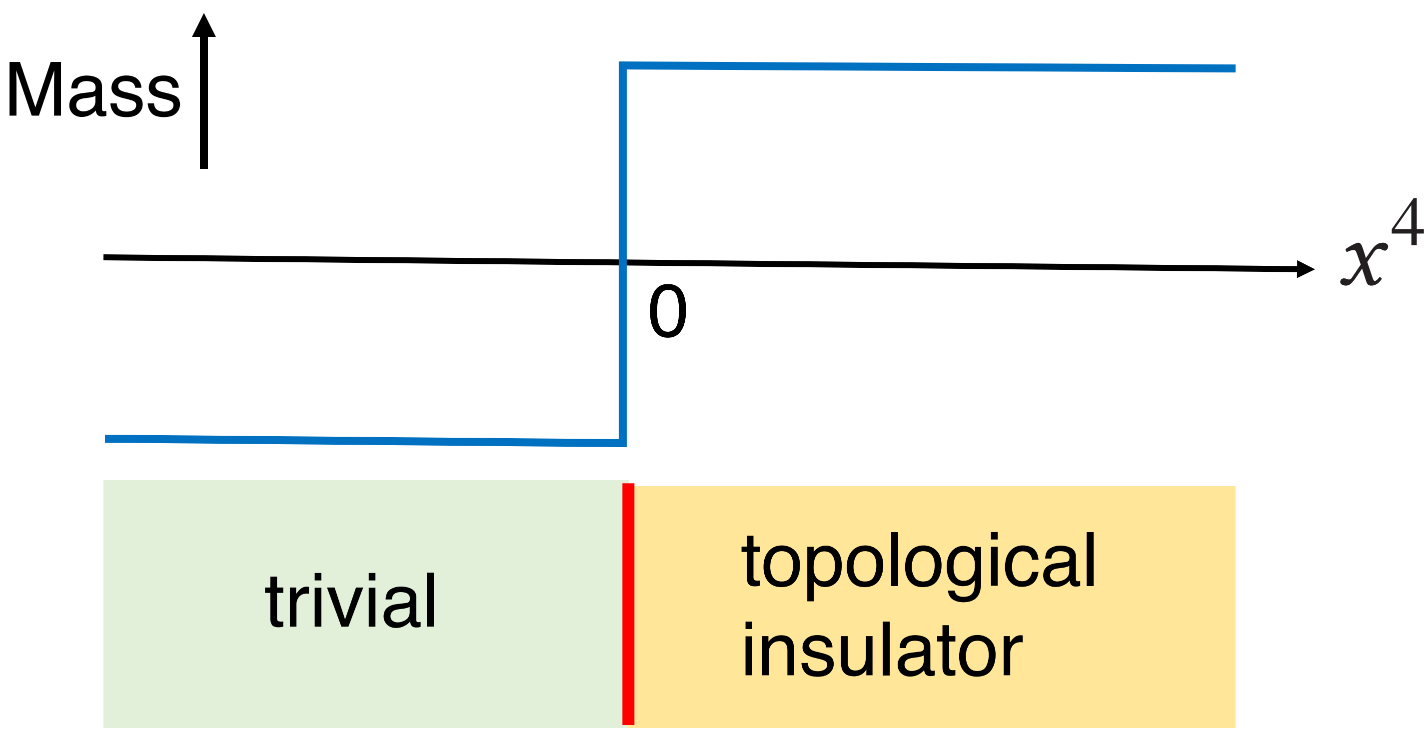
However APS setup and topological insulator look quite different!

APS		**Top. Ins.**
Massless fermion	↔	Massive fermion
Non-local B.C.	↔	Local B.C.
No massless edge localized mode	↔	Massless edge localized mode exists

We want to clarify this relation starting from a setup more close to topological insulator.

Domain-wall setup

(interface, defect, …)



In this setup, we define “domain-wall index” $\mathcal{I}$ and show

- Partition function is written as

$$Z = |Z|(-1)^{\mathcal{I}}$$

- Calculate $\mathcal{I}$ using Fujikawa’s method and obtain

$$\mathcal{I} = \frac{\eta(iD^{3D})}{2}|_{x^4=0} + \frac{1}{32\pi^2}\int_{x^4>0} d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

(= APS index)

Massive fermion and index without boundary

Massive fermion in 4 dim coupled to background gauge field

$$D := \gamma^\mu(\partial_\mu + iA_\mu)$$

$$S = \int d^4x \bar{\psi}(D + M)\psi \qquad\qquad M > 0$$

Consider the phase of the partition function

$$Z = \int D\psi D\bar{\psi} e^{-S} = \det(D + M)$$

But this is divergent …

We employ Pauli-Villars

$$Z = \frac{\det(D+M)}{\det(D-\Lambda)} \qquad \Lambda > 0$$

CP symmetry $\Rightarrow$ Z is real.

$$Z = |Z|(-1)^J$$

Let us find J

$$Z = \frac{\det(D + M)}{\det(D - \Lambda)}$$

$$= \frac{\det i\gamma_5(D + M)}{\det i\gamma_5(D - \Lambda)} = \frac{\det iH}{\det iH_{PV}}$$

$$H := \gamma_5(D + M) \qquad H_{PV} := \gamma_5(D - \Lambda)$$

Both of them are Hermitian operators.

$$\det iH = \prod_{\lambda:\text{eigenvalues of} H} i\lambda$$

$$H := \gamma_5(D+M) \qquad H_{PV} := \gamma_5(D-\Lambda) \qquad Z = \frac{\det iH}{\det iH_{PV}}$$

$$\det iH = \prod_{\lambda:\text{eigenvalues of} H} i\lambda$$

$$i\lambda = |\lambda| \exp(i\frac{\pi}{2}\mathrm{sign}\lambda)$$

$$\det iH = |\det iH| \exp(i\frac{\pi}{2}\boxed{\sum_{\lambda}\mathrm{sign}\lambda})$$

$$\eta(H) := \sum_{\lambda}\mathrm{sign}\lambda$$

regularized by zeta function regularization

"Eta invariant"

$$Z = |Z|(-1)^J$$

$$J = \frac{\eta(H)}{2} - \frac{\eta(H_{PV})}{2}$$

We can calculate $\eta(H)$ by eg. Fujikawa's method.

$$\eta(H) = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

$$= \mathrm{Ind}(D)$$

※ This is independent of M as far as M>0

※ We can also obtain

$$\eta(H_{PV}) = -\frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

For topological insulator

$$Z = |Z|(-1)^J$$

$$J = \frac{\eta(H)}{2} - \frac{\eta(H_{PV})}{2} = \eta(H) = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

$$= \mathrm{Ind}(D)$$

For trivial phase $M \longrightarrow -M$

$$Z = \frac{\det(D-M)}{\det(D-\Lambda)} = |Z|(-1)^J$$

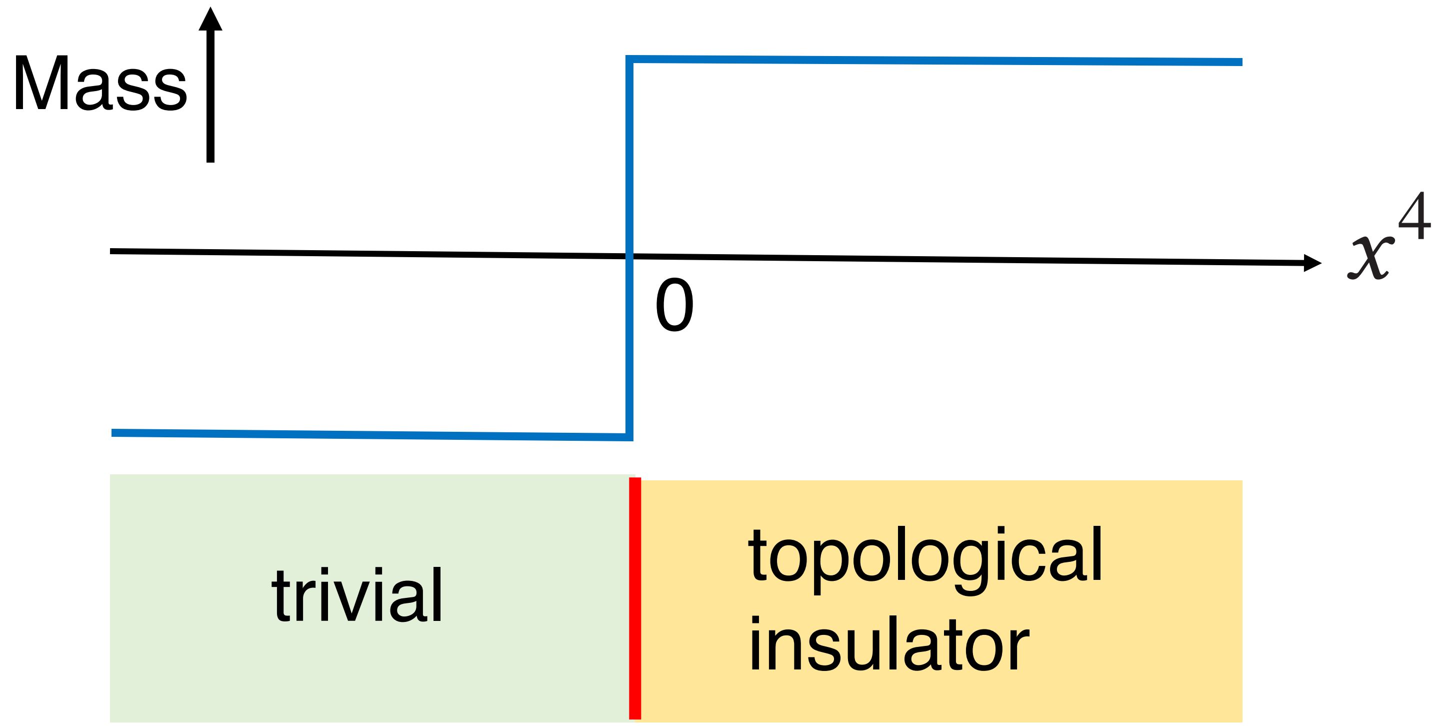
$$J = \frac{\eta(\gamma_5(D-M))}{2} - \frac{\eta(H_{PV})}{2} = 0$$

Summary

$$Z = |Z|(-1)^J$$

Index appear in the phase of the partition function of a massive fermion.

Domain-wall Dirac operator



$$S = \int d^4x \bar{\psi}(D + M\epsilon(x^4))\psi$$

$$\epsilon(x^4) = \begin{cases} -1, & (x^4 < 0) \\ +1, & (x^4 > 0) \end{cases}$$

By the same argument

$$Z = \frac{\det(D + M\epsilon(x^4))}{\det(D - \Lambda)} = |Z|(-1)^{\mathcal{I}}$$

$$\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

$$H_{DW} := \gamma_5(D + M\epsilon(x^4))$$

Let us call this integer “domain-wall index”

We calculated $\eta(H_{DW})$ by Fujikawa's method.

Result:

$$\eta(H_{DW}) = \underline{\eta(iD^{3D})|_{x^4=0}} + \frac{1}{32\pi^2}\int d^4x \underline{\epsilon(x^4)}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

Domain-wall index

$$\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$$

$$= \frac{\eta(iD^{3D})}{2}|_{x^4=0} + \frac{1}{32\pi^2}\int_{x^4>0} d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

=(APS index in $x^4 > 0$)

Comments

- $\eta(H_{DW})$ is independent of M as far as $M > 0$

- Honestly speaking, we have shown

$$\eta(H_{DW}) = f(A)|_{x^4=0} + \frac{1}{32\pi^2}\int d^4x \epsilon(x^4)\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

s.t.

$\delta f(A) = \frac{\delta CS(A)}{\pi}$ for arbitrary variation δA_μ

$f(A) = \eta(iD^{3D})$ satisfy this property.

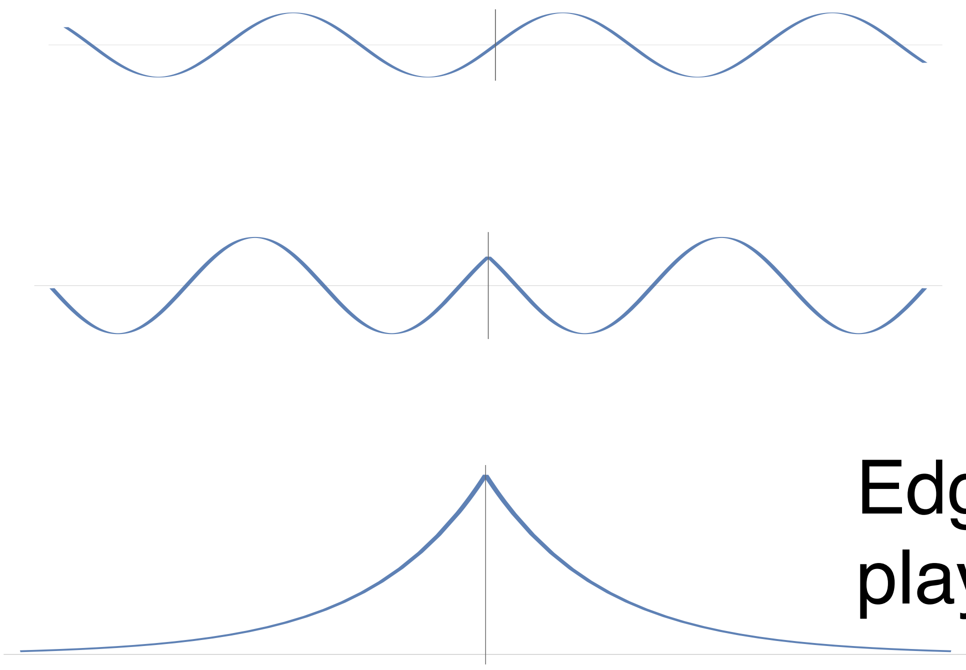
We could not give a complete proof of $f(A) = \eta(iD^{3D})$ although there are some evidences.

Comments

In our calculation by Fujikawa's method, we inserted complete basis.

We choose eigen functions of $-\partial^2 + M^2 + M\gamma_4\delta(x_4)$

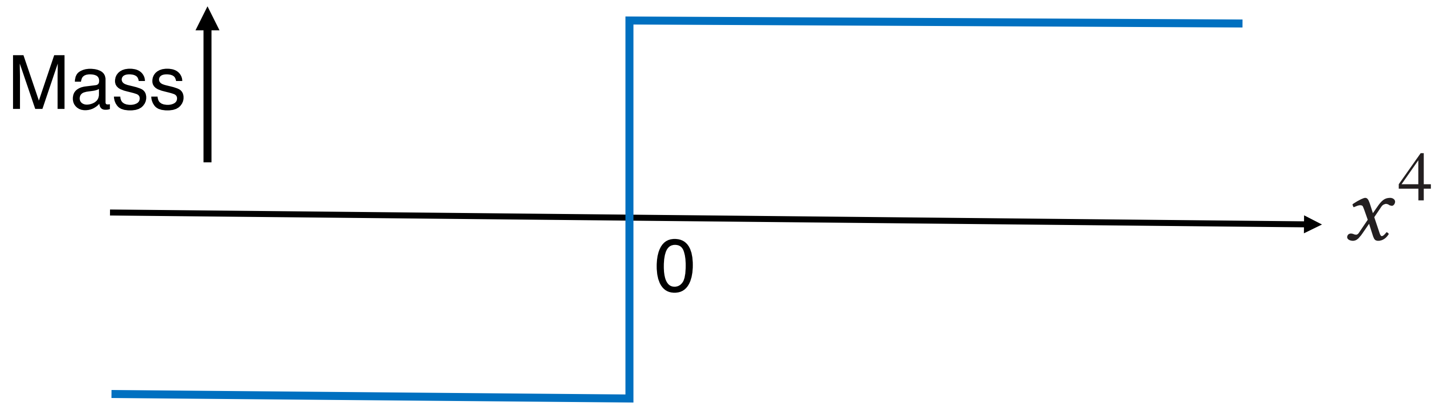
which appears in H_{DW}^2



Edge localized modes play a crucial role.

Summary

Domain-wall setup (close to topological insulator)



- We define domain-wall index $\mathcal{I} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}$
- It appears in the phase of the partition function
- We have shown by the explicit calculation that (domain-wall index) = (APS index)

Future problem

- Give a complete proof.

  The difficulty seems to come from the non-locality of $\eta(iD^{3D})$

- Including background metric, other dimensions,…