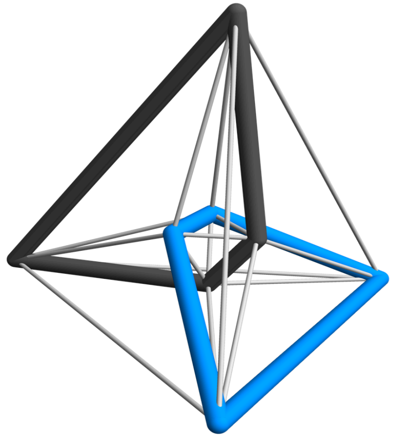
Non-invertible topological defects in 4-dimensional Z_2 pure lattice gauge theory

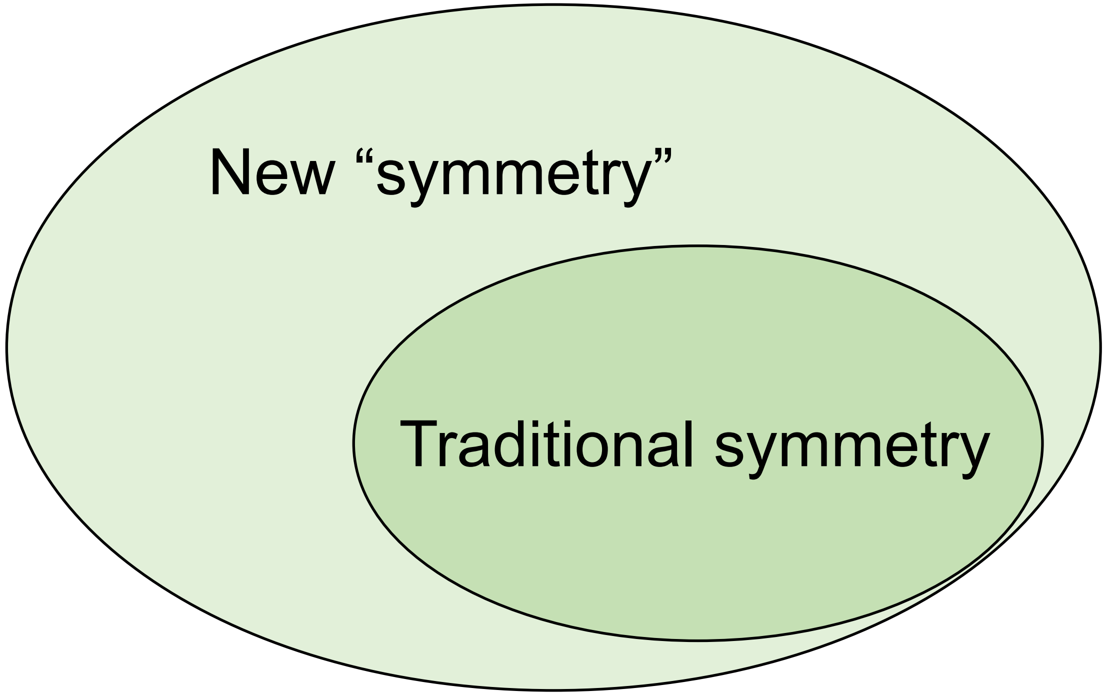


Satoshi Yamaguchi (Osaka University)

Based on
M. Koide, Y. Nagoya, SY, arXiv:2109.05992, to appear in PTEP

Introduction

Concept of symmetry is changing.



Generalized symmetry plays an important role in

- Phase structure of QFT

  [Gaiotto, Kapustin, Seiberg, Willet 14], [Gaiotto, Kapustin, Komargodski, Seiberg 17],...

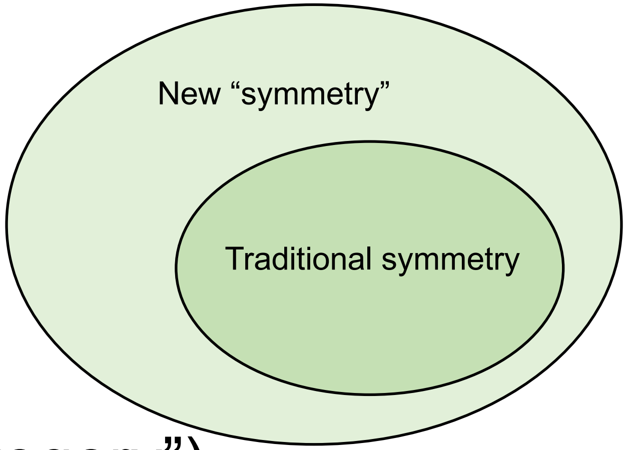
- String theory

  ...[Bergman, Tachikawa, Zafrir 20], [Bah, Bonetti, Minasian 20],
  [Morrison, Schafer-Nameki 20], [Albertini, del Zotto, Garcia Etxebarria, Hosseini 20],
  [Apruzzi, Dierigl, Lin 20], [Benetti Genolini, Tizzano 21], [Apruzzi, van Beest, Gould, Schafer-Nameki 21]...

- Physics beyond the standard model (“naturalness”)

Non-invertible symmetry = a class of new “symmetry”

No group structure



It is proved to be useful at least in 2 dimensions (“fusion category”)

[Verlinde 88], [Moore, Seiberg 88, 89], [Frohilich, Fuchs, Runkel, Schwigert 02--06], [Bhardwaj-Tachikawa 17], [Chang, Lin, Shao, Yin 18], [Komargodski, Ohmori, Roumpedakis, Seifmashri 20]...

Examples in 3 or higher (in particular 4) dimensions ?

[Ji, Wen 19], [Kong, Lan, Wen, Zhang, Zheng 20], [Rudelius, Shao 20], [Heidenreich et. al. 21], [Nguyen, Tanizaki, Unsal 21],...
[Johnson-Freyd 20]

We find an example of non-invertible symmetry in 4 dimensions

[Koide, Nagoya, SY 21]

4-dimensional Z_2 lattice gauge theory

Duality [Wegner 71]

1-form Z_2 center symmetry

Non-invertible symmetry

[Koide, Nagoya, SY 21]

Duality [Wegner 71]

1-form Z_2 center symmetry

Non-invertible symmetry

We also find the “algebra” of this non-invertible symmetry.

This symmetry will not be only a special symmetry of a special theory, but it appears in many theories (we expect).

Important notion [Gaiotto, Kapustin, Seiberg, Willet 14]

Topological defects

Plan:

- Symmetry ⇒ topological defect
- Generalized symmetry
- Topological defects in 4d Z_2 lattice gauge theory — overview—
- Topological defects in 4d Z_2 lattice gauge theory — detail—
- Summary and discussion

Symmetry ⇒ topological defect

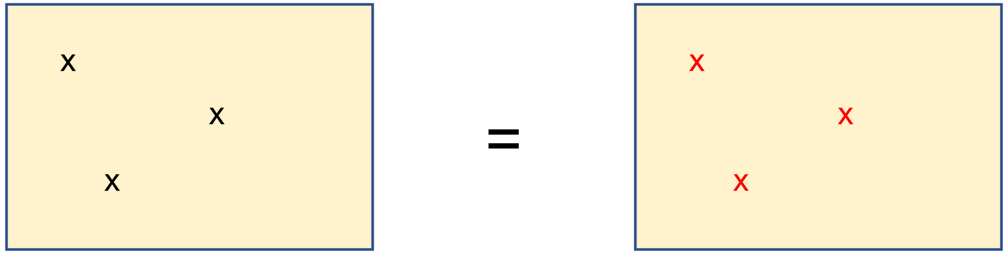
Symmetry

transformation $\phi(x) \to \phi'(x)$ $S[\phi'] = S[\phi]$

Relations between correlation functions

(Even if you do not know the action, you can start from here.)

1 Global form



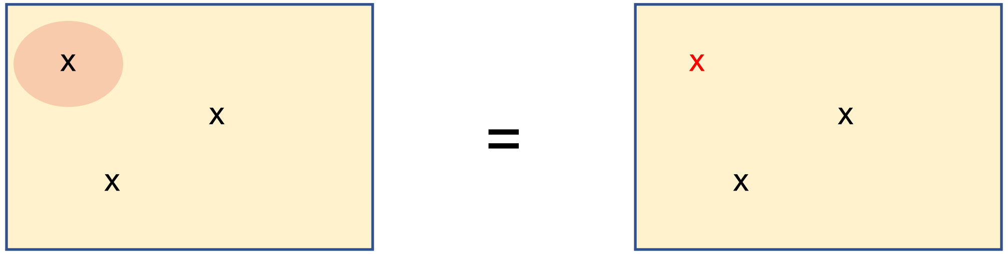
$$\langle \mathcal{O}_1'(x_1)\mathcal{O}_2'(x_2)\cdots\rangle = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\rangle$$

Good: Any group symmetry (continuous, discrete or disconnected)

Bad:

- Global (You cannot forget something far away from you)
- Not valid when SSB occurs.

2 Local form



Ward-Takahashi identity

$$\langle \epsilon \partial_\mu J^\mu(x) \mathcal{O}_1(x_1) \cdots \rangle = \delta(x - x_1) \langle \delta \mathcal{O}_1(x_1) \cdots \rangle$$

infinitesimal parameter

if x does not coincide with the points where other operators are inserted

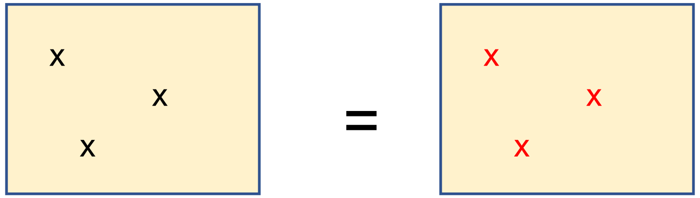
Good:

- Local (You can forget something far away from you)
- Valid even when SSB occurs.

Bad: only for infinitesimal transformation

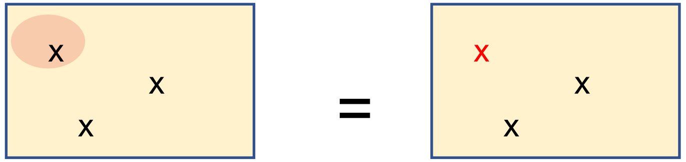
1 Global form

$$\langle \mathcal{O}_1'(x_1)\mathcal{O}_2'(x_2)\cdots\rangle = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\rangle$$



2 Local form

$$\langle \epsilon\partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\cdots\rangle = \delta(x-x_1)\langle \delta\mathcal{O}_1(x_1)\cdots\rangle$$



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2 is much more convenient than 1

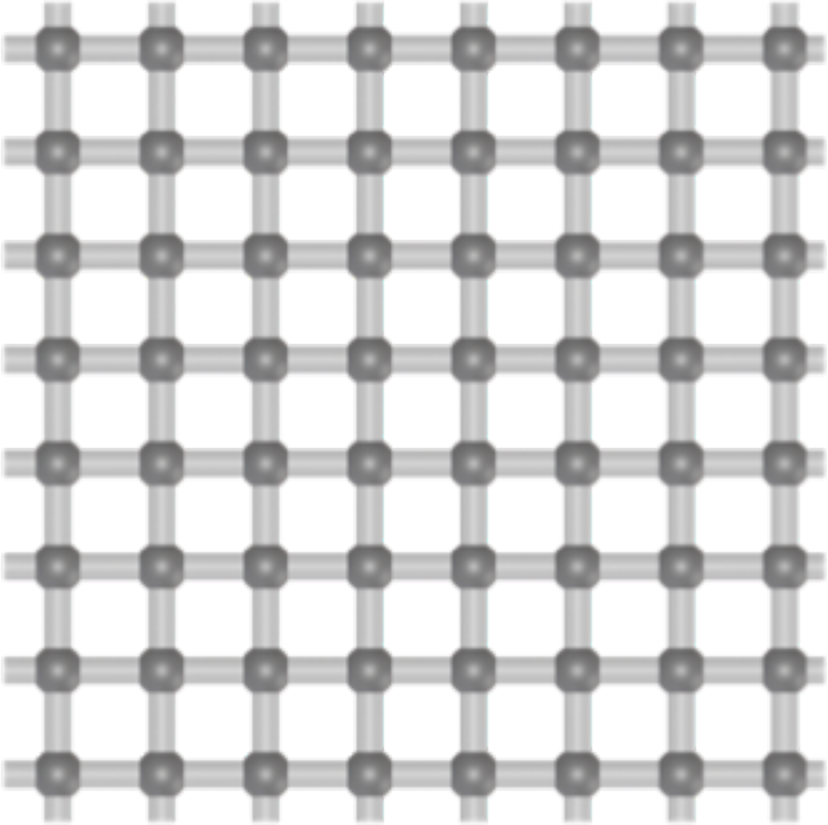
Any local form for discrete or disconnected group symmetry?

Let's try!

Example: Ising model

$$\sigma(x) = \pm 1$$

x: label of a site.



$$Z = \sum_{\{\sigma\}} \exp(-S(\sigma))$$

$$S(\sigma) = -K \sum_{\langle xy \rangle:\text{links}} \sigma(x)\sigma(y)$$

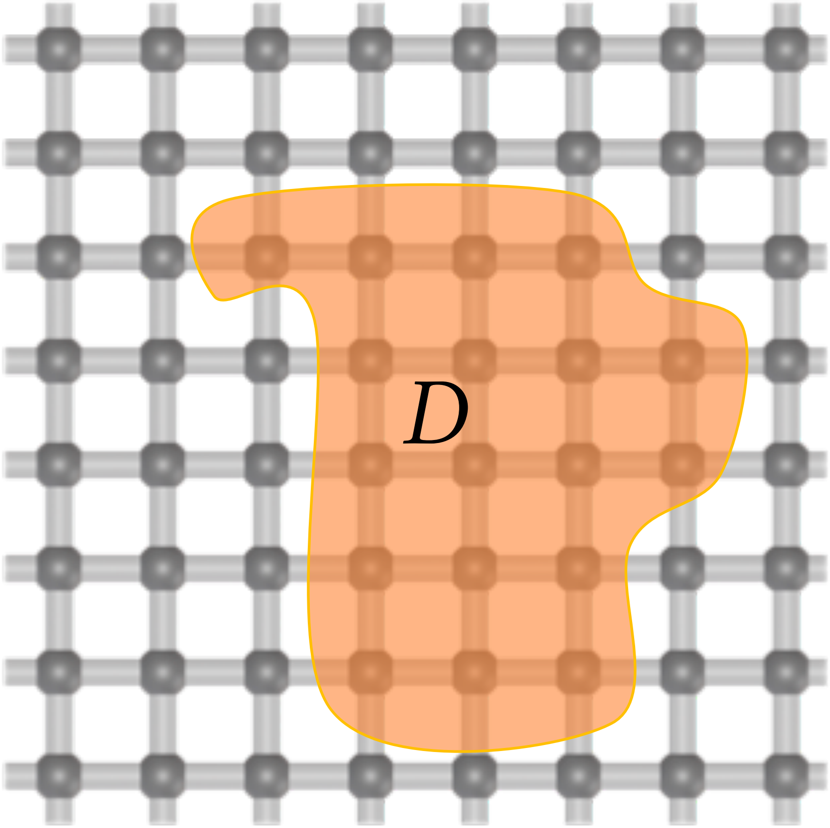
Z_2 symmetry

$$\sigma(x) \rightarrow \sigma'(x) = -\sigma(x)$$

Spacetime dependent transformation

region D

$$\sigma'(x) = \begin{cases} -\sigma(x) & (x \in D) \\ \sigma(x) & (x \notin D) \end{cases}$$



$$\sigma'(x) = \begin{cases} -\sigma(x) & (x \in D) \\ \sigma(x) & (x \notin D) \end{cases}$$

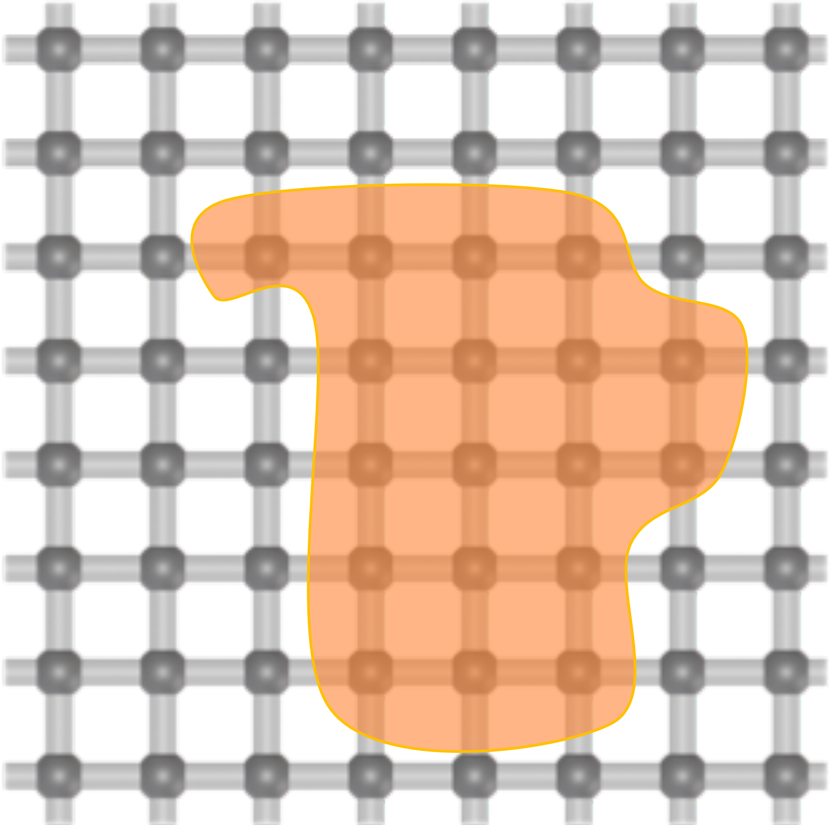
$$Z = \sum_{\{\sigma\}} \exp(-S(\sigma))$$

just change the letter

$$= \sum_{\{\sigma'\}} \exp(-S(\sigma'))$$

use this

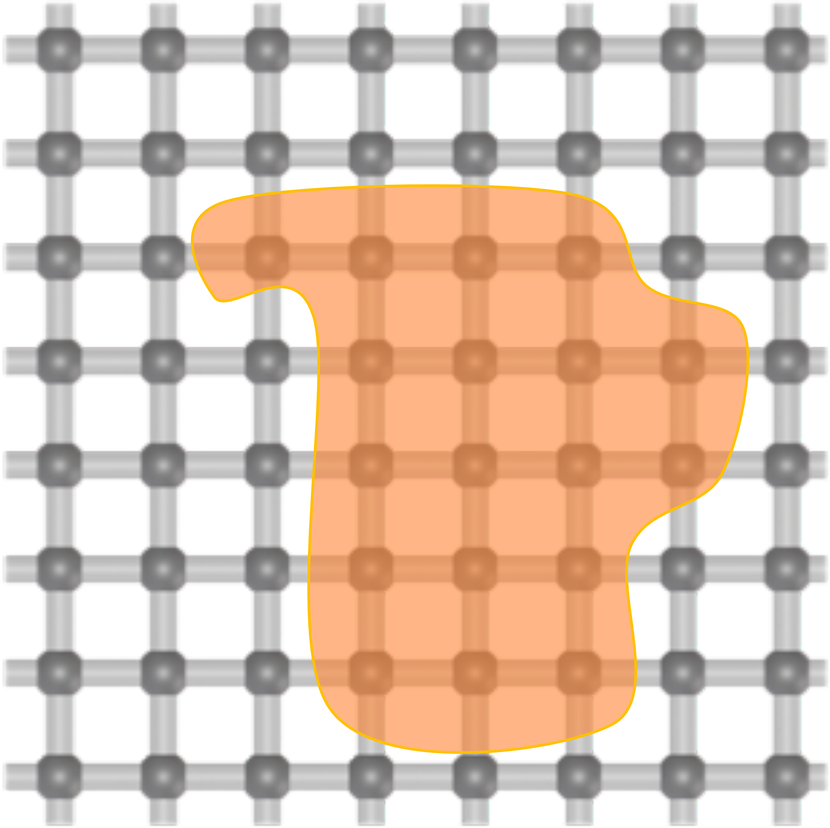
$$= \sum_{\{\sigma\}} \exp(-S_D(\sigma))$$

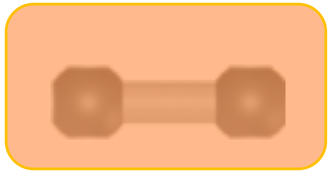


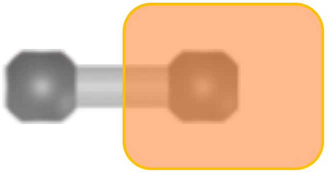
$$S_D(\sigma) \neq S(\sigma)$$

How different?

$$Z=\sum_{\{\sigma\}}\exp(-S(\sigma))=\sum_{\{\sigma'\}}\exp(-S(\sigma'))=\sum_{\{\sigma\}}\exp(-S_D(\sigma))$$

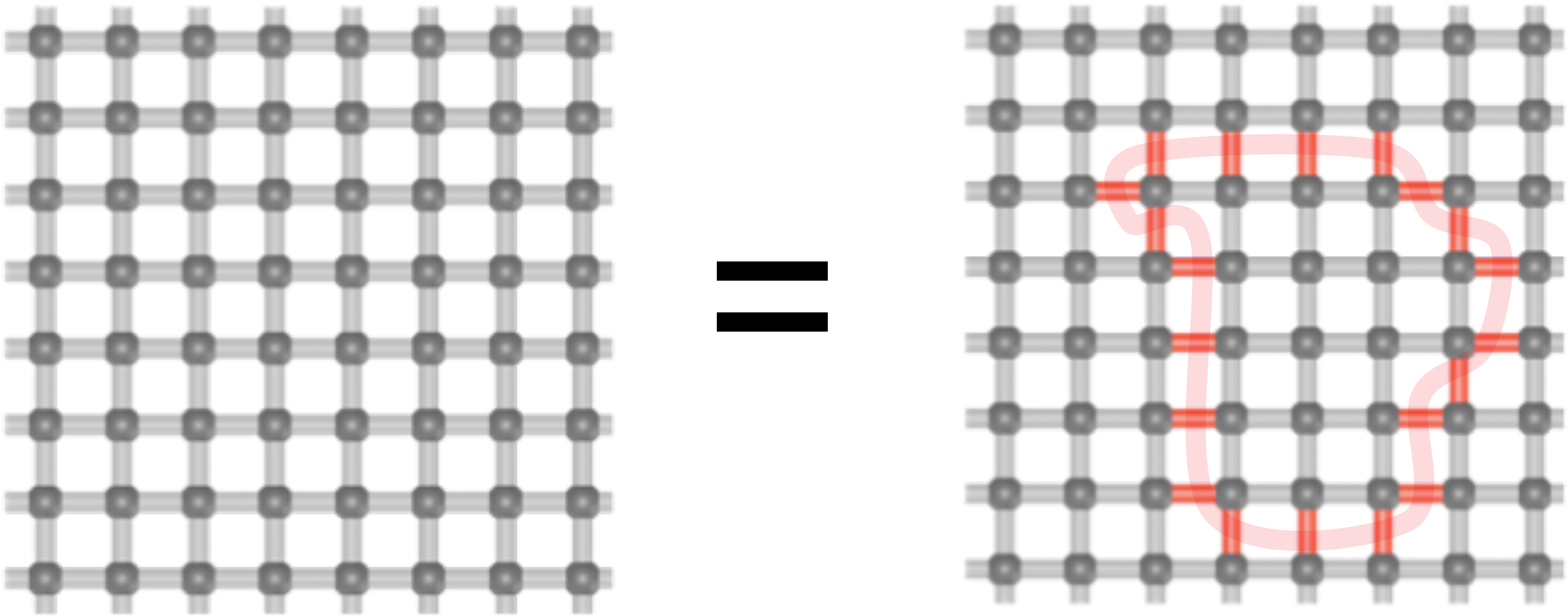


 =  $= \exp(K\sigma(x)\sigma(y))$

 $= \exp(-K\sigma(x)\sigma(y)) =:$ 

“Defect”

Such a defect associated to symmetry is called a “symmetry defect.”

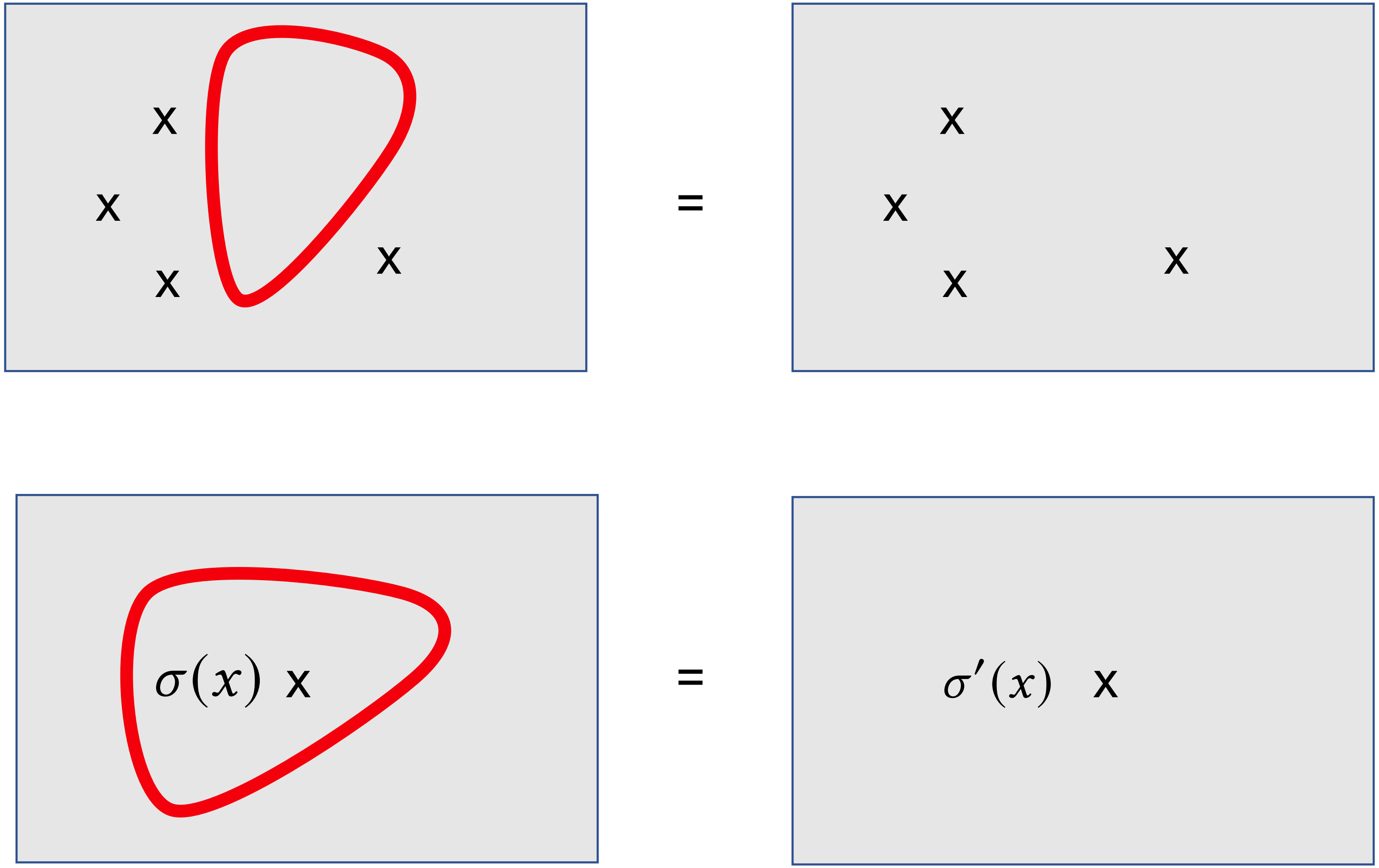


$$Z = \sum_{\{\sigma\}} \exp(-S(\sigma)) = \sum_{\{\sigma\}} \exp(-S_D(\sigma)) = \sum_{\{\sigma\}} \exp(-S(\sigma))U(\Sigma)$$

$$U(\Sigma) := \exp(-S_D(\sigma) + S(\sigma)) \qquad \Sigma := \partial D$$

$$1 = \langle U(\Sigma) \rangle$$

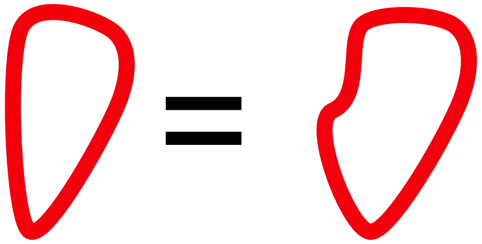
Relation to other operator



(Local form of) Ward-Takahashi identity

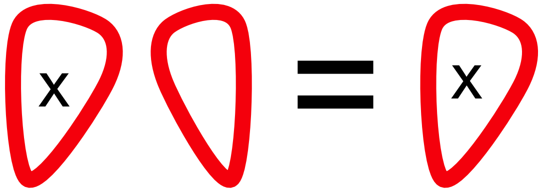
Ordinary symmetry defect

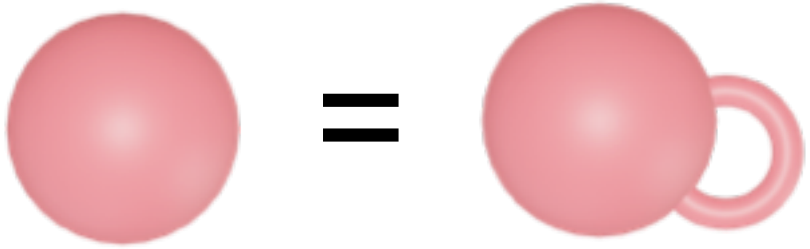
- Codimension 1
- Topological



- Invertible

The expectation value is the same if the difference is the boundary of a region without any insertion.

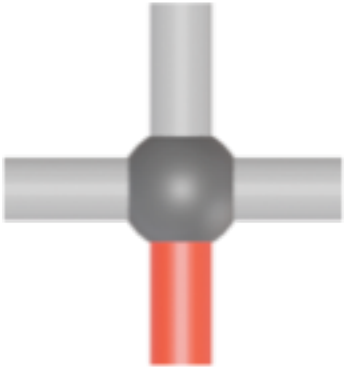




Technical remark

There are many ways to realize symmetry defect.

Eg.



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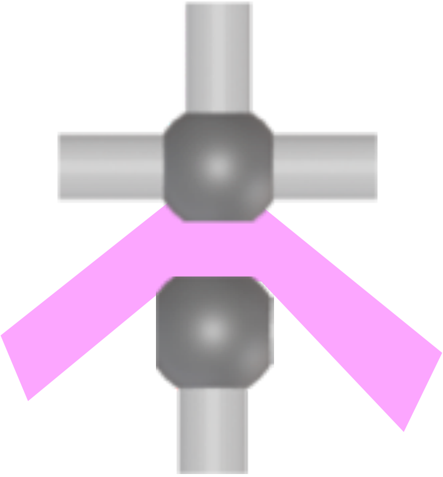
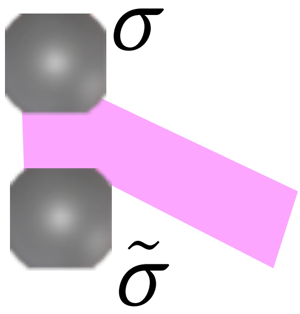
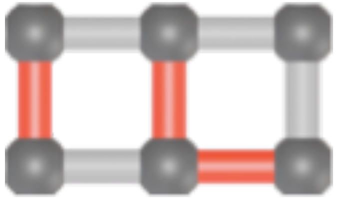




$= \delta_{\sigma,-\tilde{\sigma}}$

(It does not include any information of the action)



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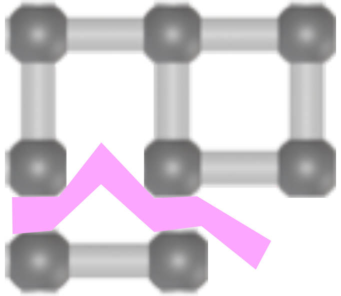

Plan:

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Generalized symmetry

So far

Symmetry ⟹ **Topological defect**

How about this way? ⟸

Not always. But WT-like identity exists for an arbitrary topological defect.

Topological defects should be able to be used in a similar way to symmetry!

Ordinary symmetry topological defect

- Codimension 1
- Invertible

Generalize

Codimension q+1: q-form symmetry

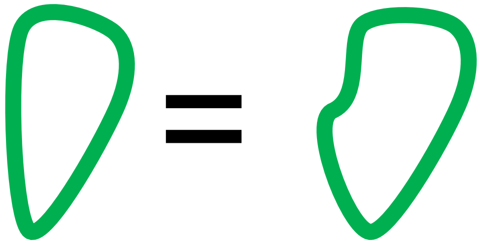
Non-invertible:

Non-invertible symmetry

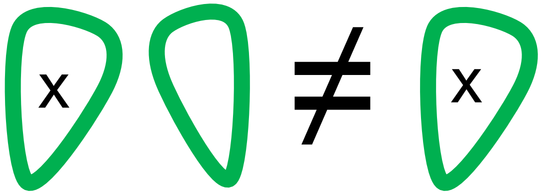
(General topological defect)

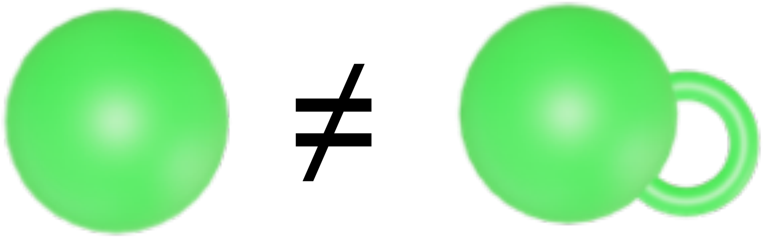
Non-invertible symmetry

- Topological

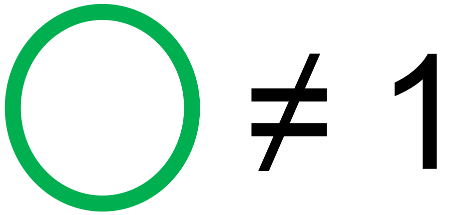


- Non-invertible





In particular



Example:

A lot of examples in 2 dimensions. Eg. Verlinde line in rational conformal field theory.

It is actually useful to investigate the phase structure of two dimensional quantum field theories.

[Chang, Lin, Shao, Yin 18], [Komargodski, Ohmori, Roumpedakis, Seifmashri 20], [Nguyen, Tanizaki, Unsal 21],...

It would be nice to have such a tool in 4 dimensions.

(We do not have rational conformal field theory...)

Example:

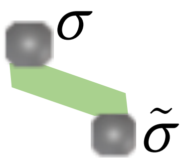
Kramers–Wannier duality in 2 dimensional Ising model.

(An example of Verlinde lines)

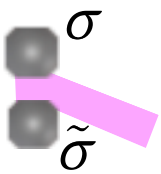
Lattice approach [Aasen, Mong, Fendley 16]

Explicitly constructed the duality defect.

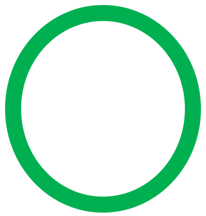
(The construction does not depend on rational CFT)



$$= \begin{cases} -1 & (\sigma = \tilde{\sigma} = -1) \\ +1 & (\text{others}) \end{cases}$$



$$= \delta_{\sigma,-\tilde{\sigma}} \qquad \cdots$$



$$= \sqrt{2} \neq 1$$

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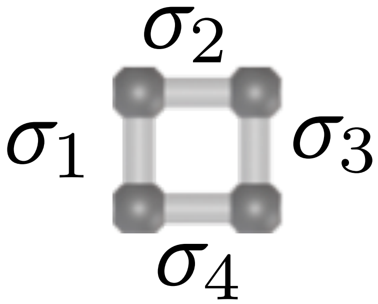
Topological defects in 4 dimensional Z_2 lattice gauge theory

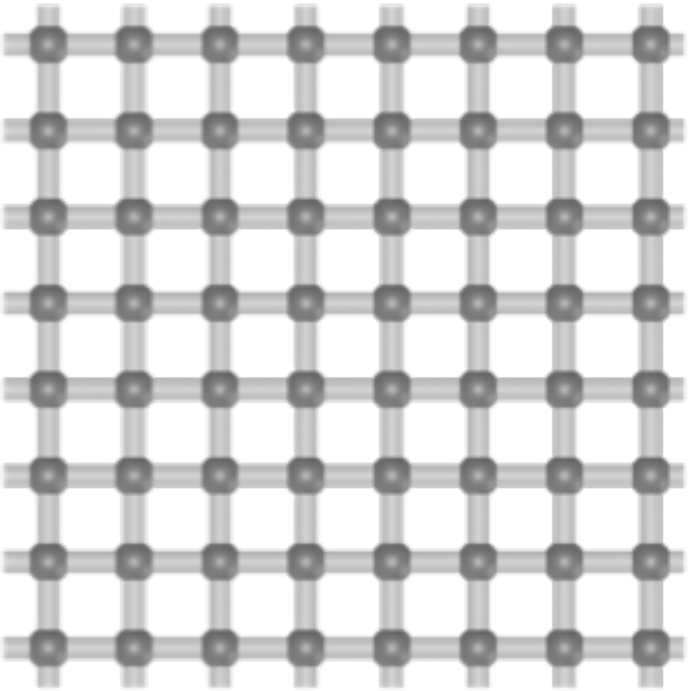
— overview —

4-dimensional Z_2 lattice gauge theory

$$Z = \sum_{\{\sigma\}} \exp\left[K \sum_{\text{all } \square} \sigma_1\sigma_2\sigma_3\sigma_4\right]$$

$$K \sim \frac{1}{g^2}$$



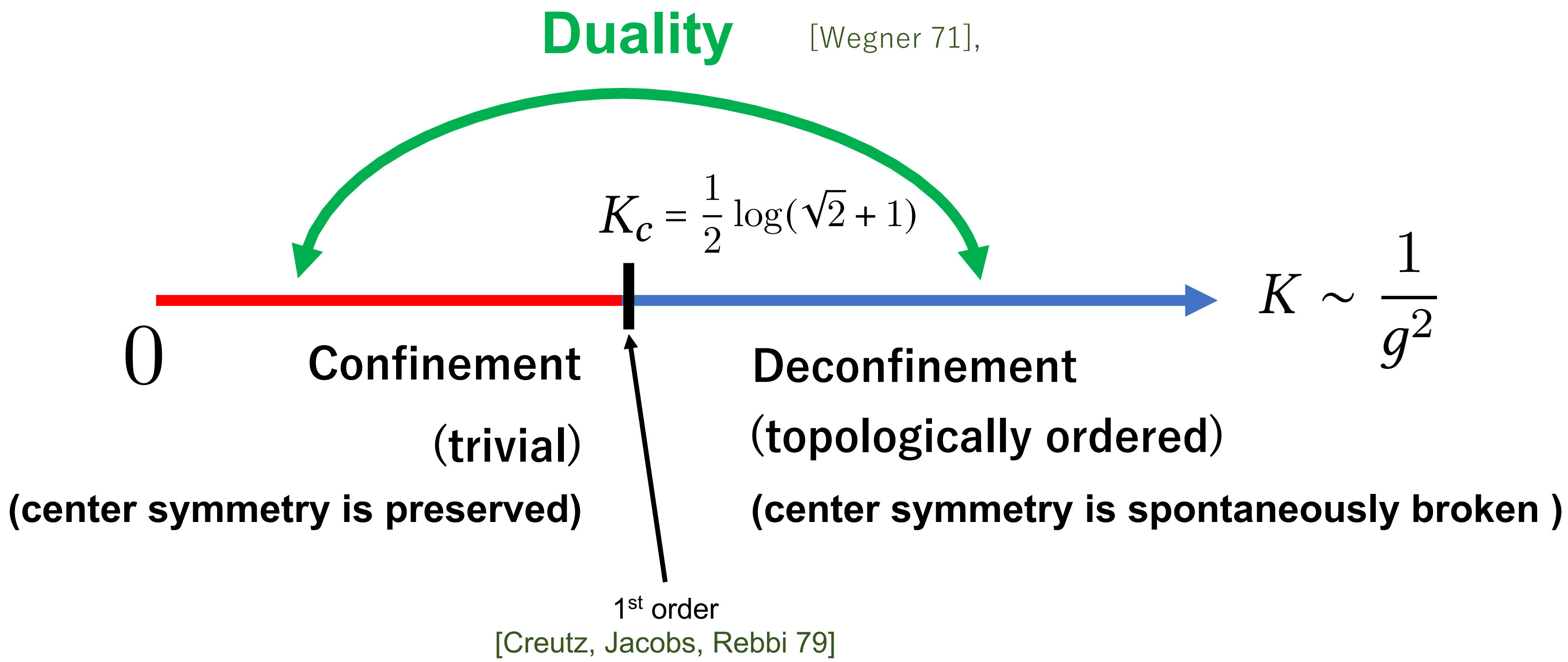


4-dimensional cubic lattice

Put a spin at each **link**

$$\sigma = \pm 1$$

4-dimensional Z_2 lattice gauge theory

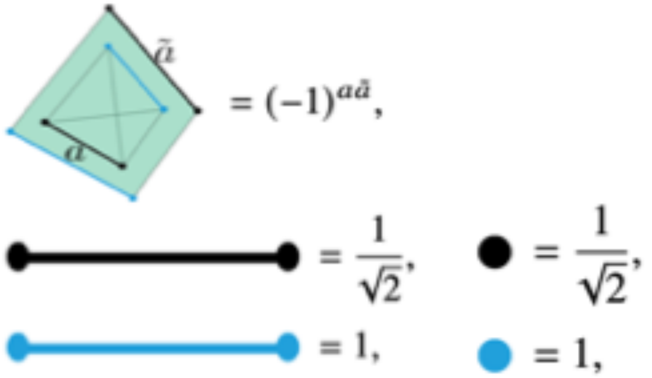


Center symmetry: 1-form Z_2 symmetry

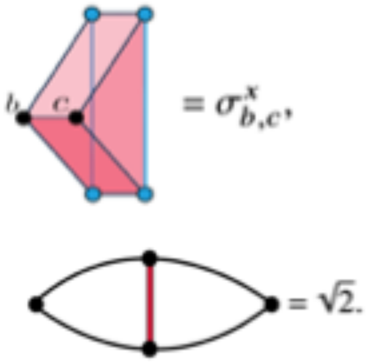
We constructed

- **topological defects corresponding to the duality and the center symmetry**

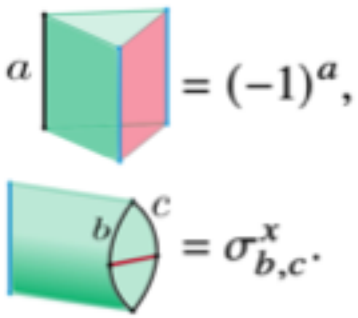
codim 1



codim 2



- **Junctions**



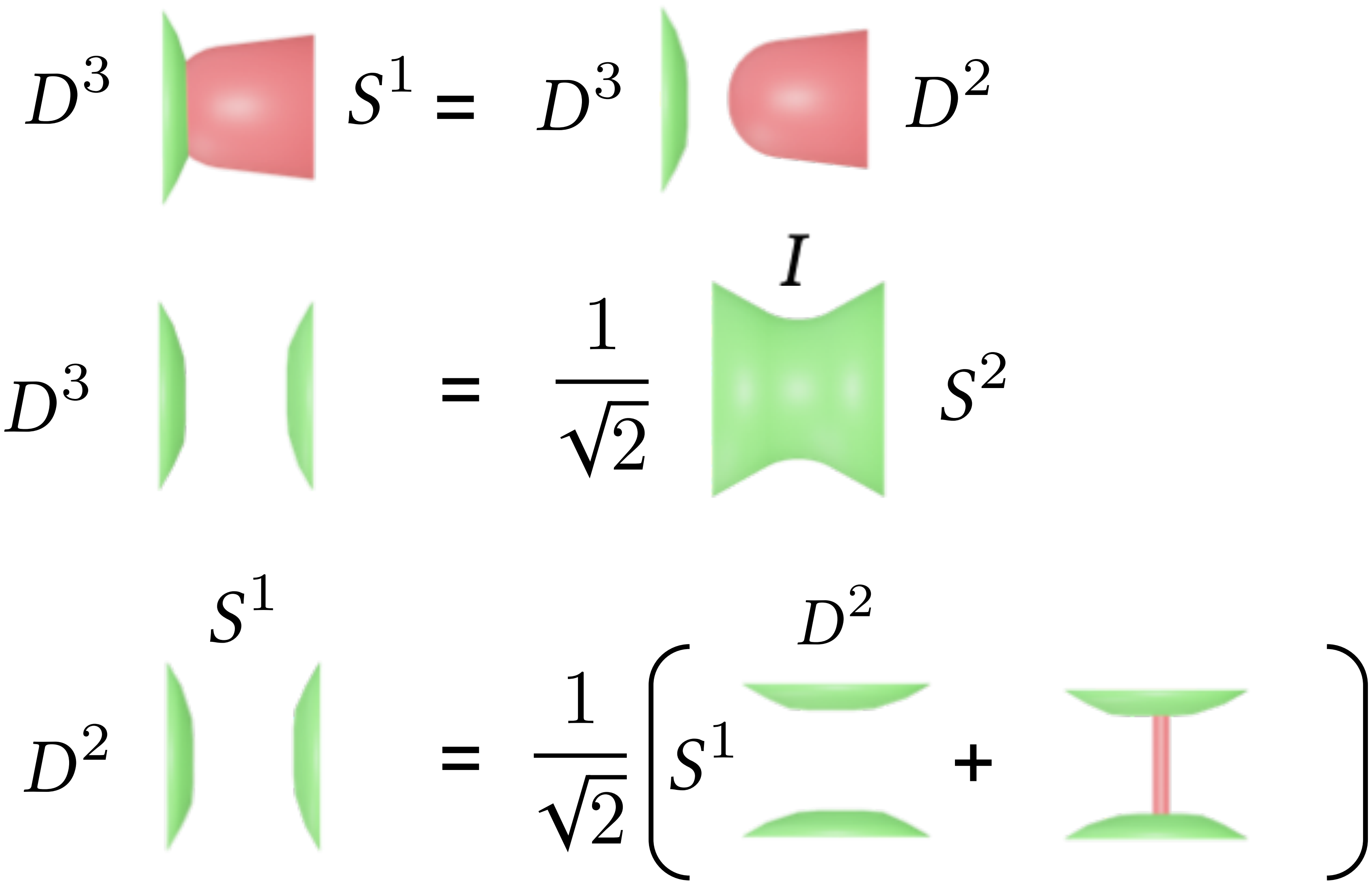
※Since we are working in lattice, these results are regorous.

We calculated

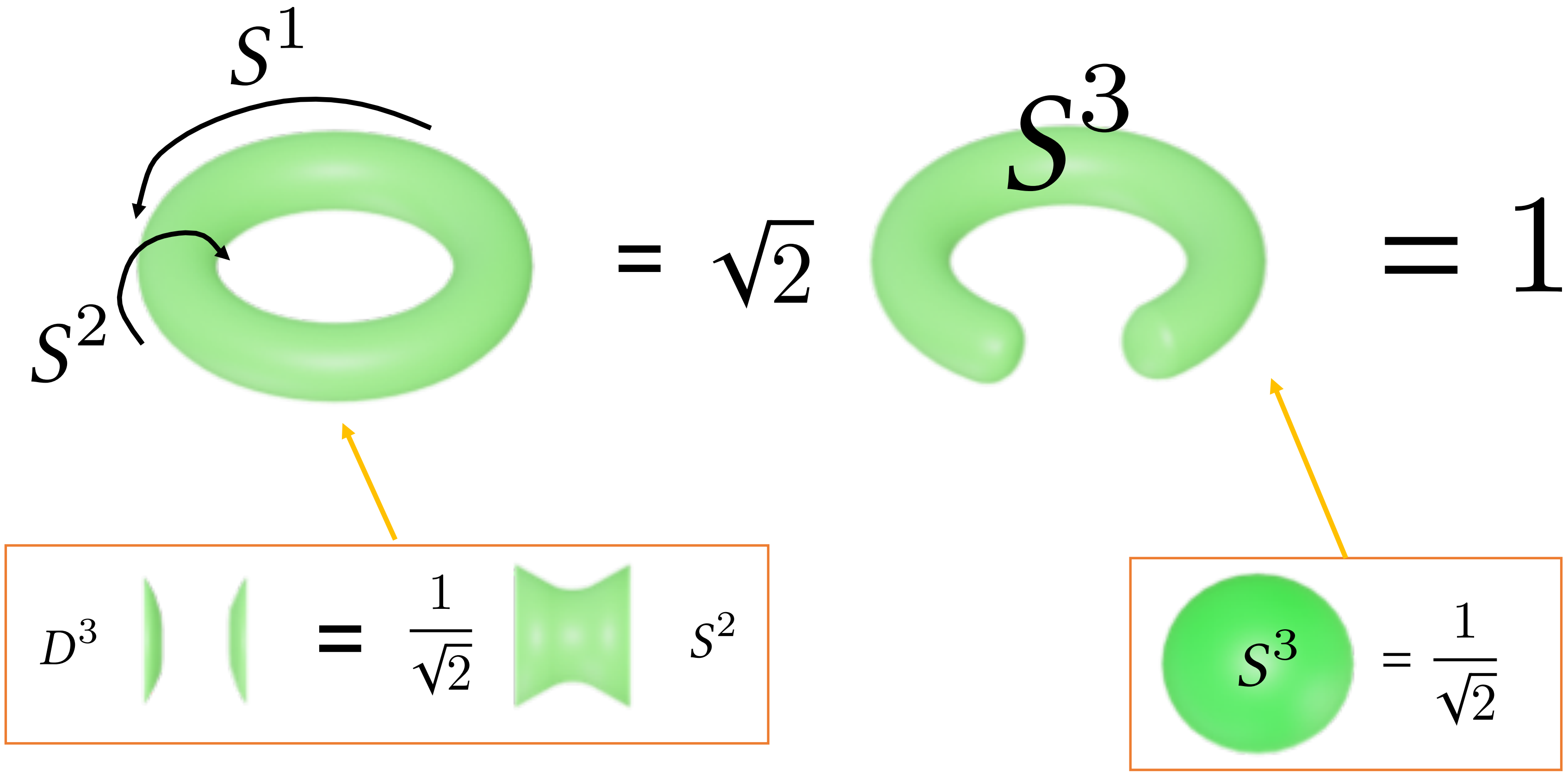
$$= \frac{1}{\sqrt{2}} \neq 1$$

The duality defect is non-invertible!

Crossing relations (algebra of the symmetry)



An example of expectation values



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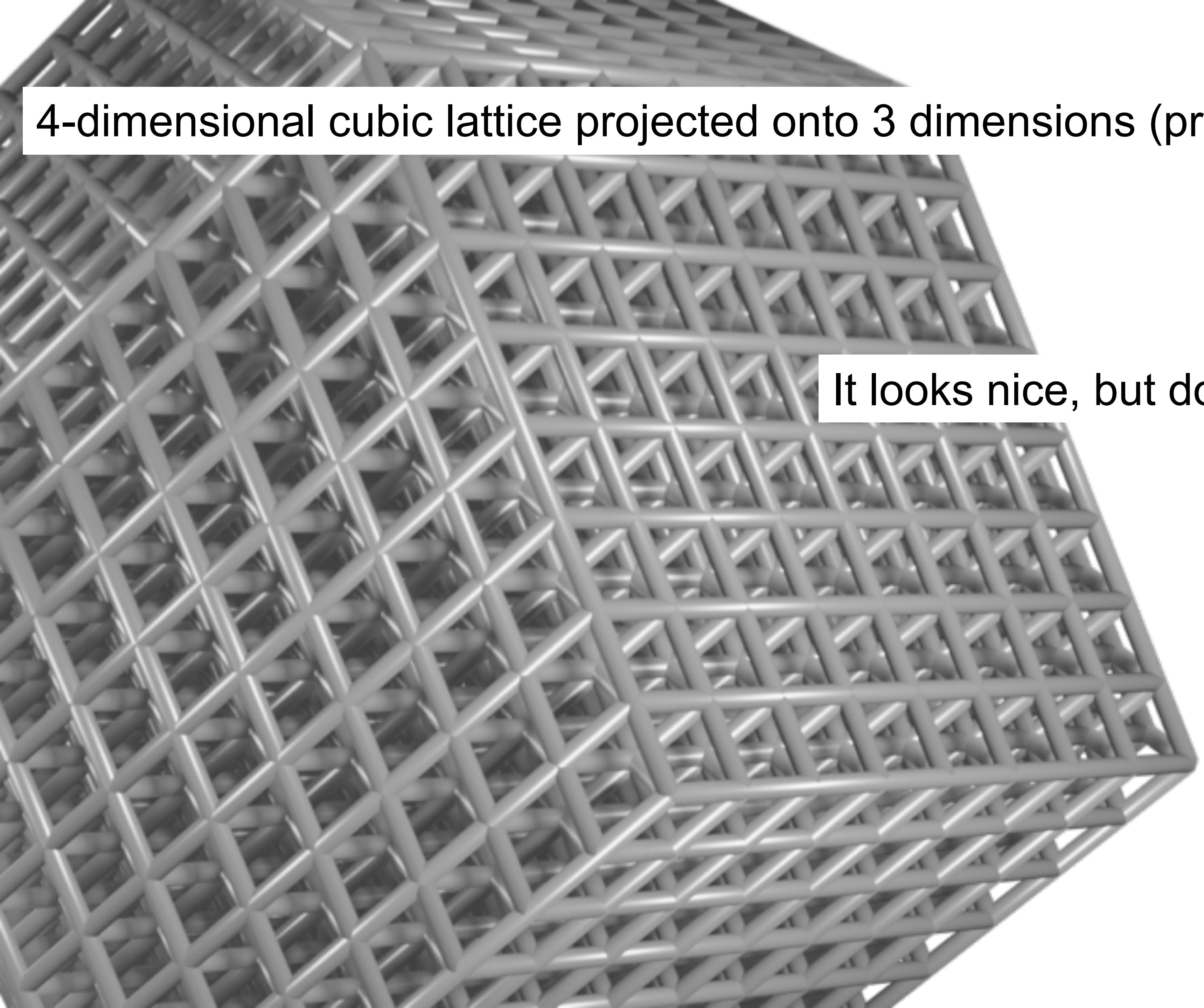
Topological defects in 4 dimensional Z_2 lattice gauge theory

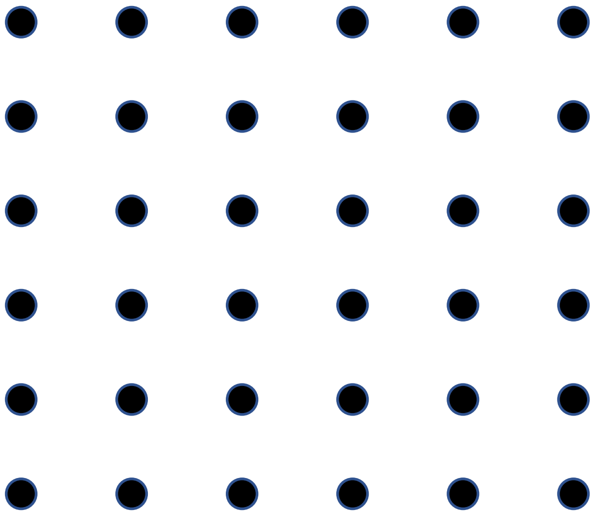
— Detail —

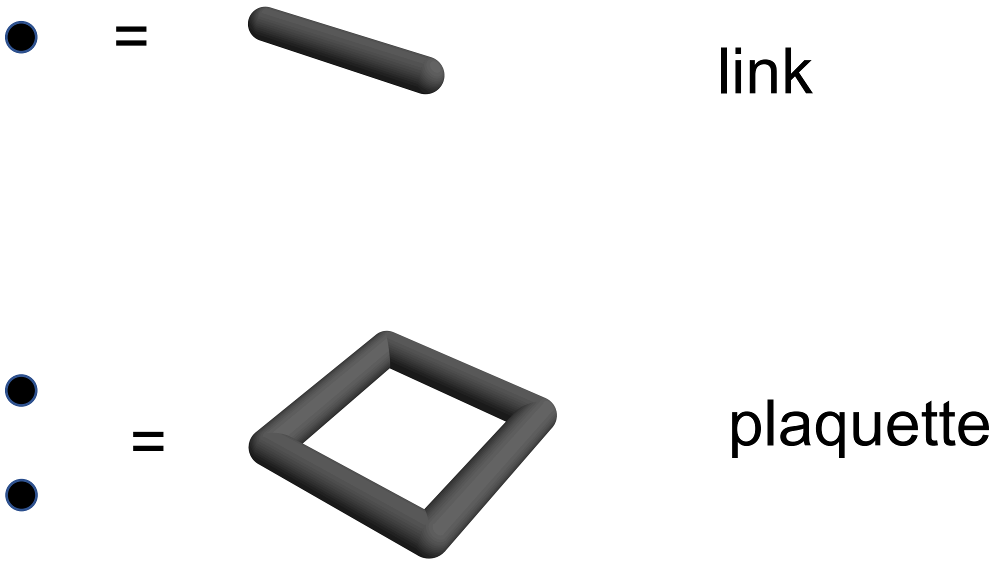
4-dimensional cubic lattice projected onto 3 dimensions (projected onto 2 dimensions)



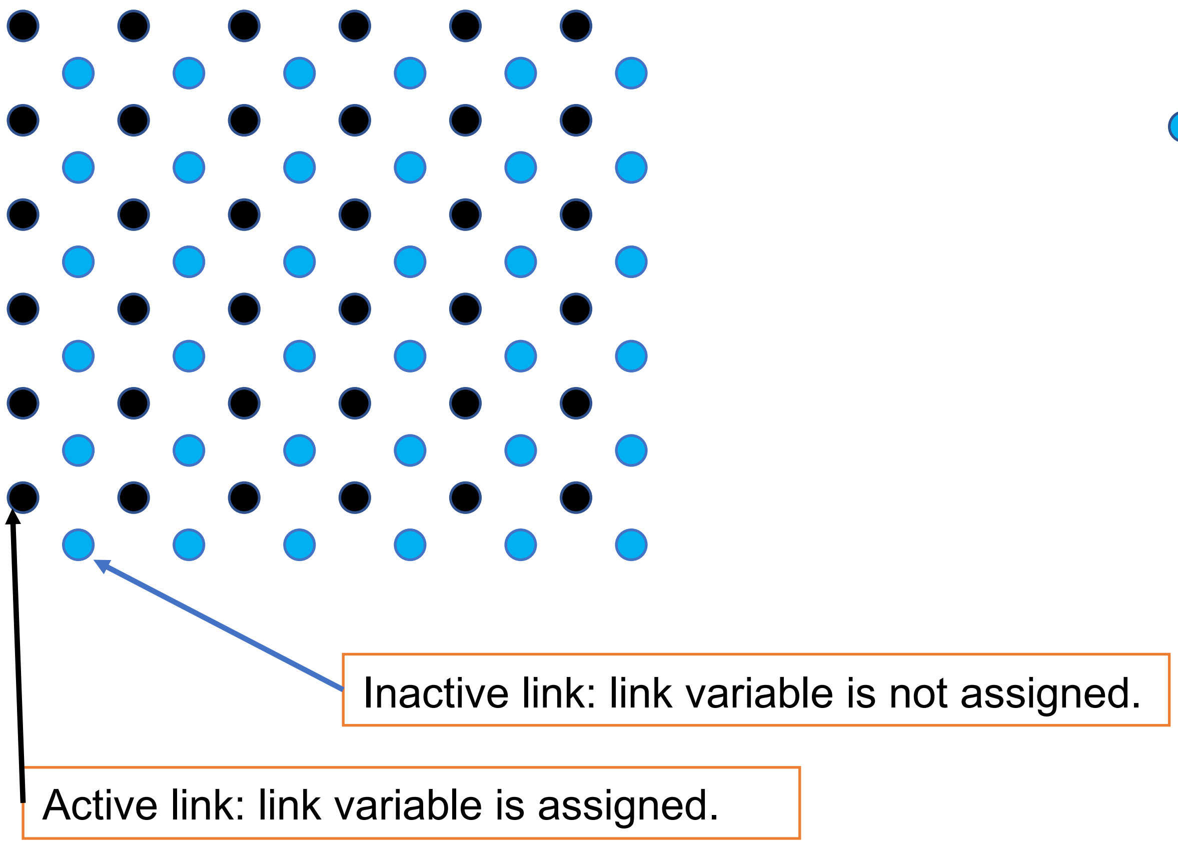
It looks nice, but does not help understanding.

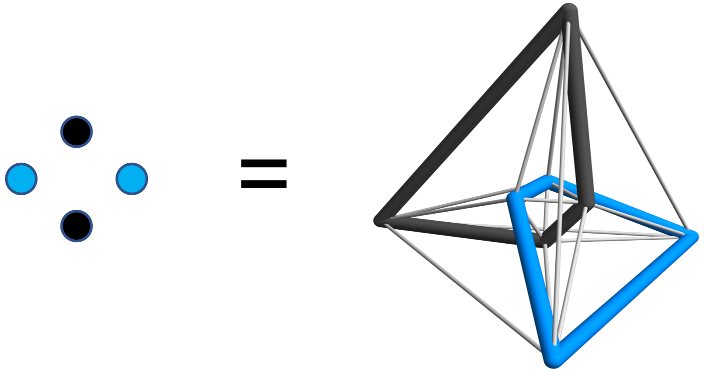
We use 2D illustration for 4D.



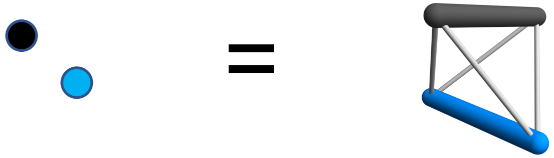


We prepare an auxiliary lattice which is dual to the original lattice.





“16-cell”



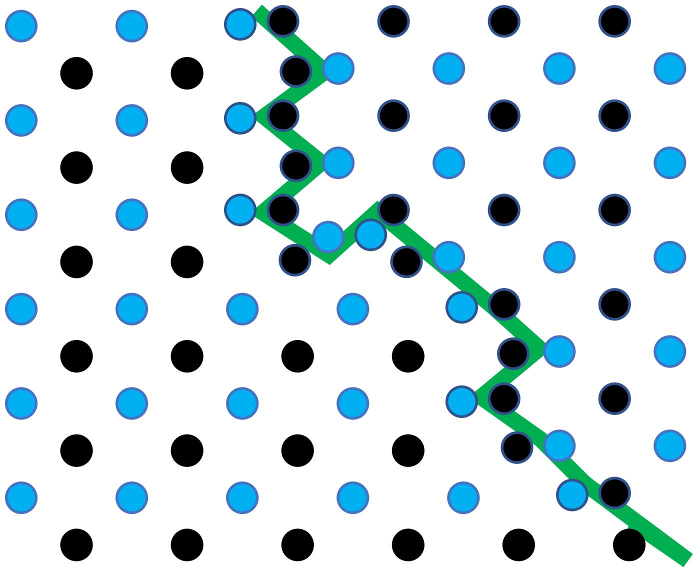
tetrahedron

We assign the Boltzmann weight for each 16-cell.
It is equivalent to assign one for each plaquette.

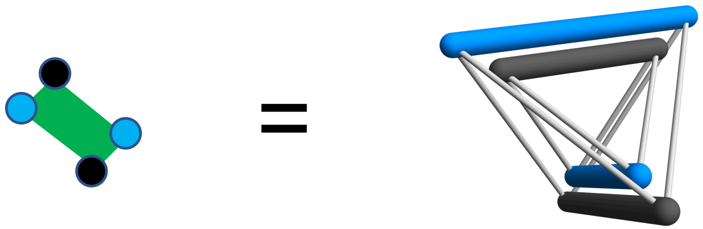
Duality defect

Double the links on the duality defect.

Exchange active and inactive links across the duality defect

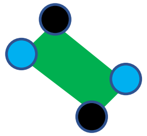


Building block



tetrahedral prism

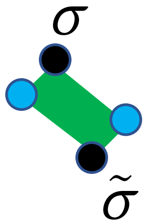
assign the weight for each building block

Assign the weight for each building block  , so that the defect is topological.

Require

Lots of such relations. → Highly overdetermined.

There is a unique physically sensible solution up to some sign conventions.

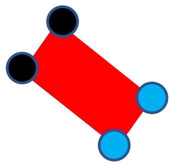
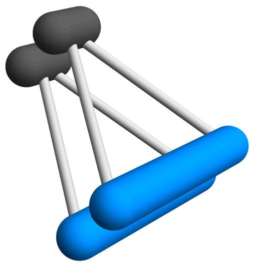


$$= \begin{cases} -1 & (\sigma = \tilde{\sigma} = -1) \\ +1 & (\text{others}) \end{cases}$$

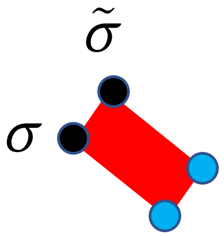
(※We also have to assgin some weights for active and inactive sites and links)

1-form Z_2 center symmetry defect

Codimension 2

 = 

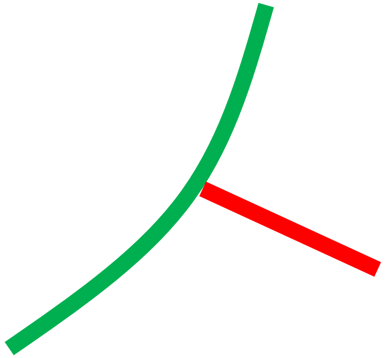
triangular prism



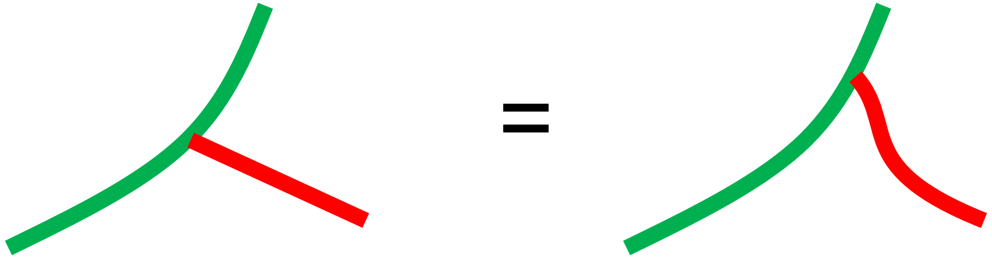
$= \delta_{\sigma,-\tilde{\sigma}}$

Topological and invertible

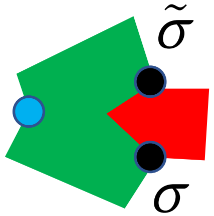
Junctions



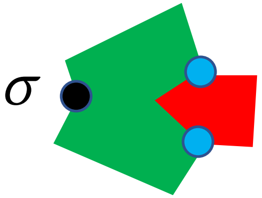
Weights are determined so that the junction is topological



Two kinds



$$= \delta_{\sigma,-\tilde{\sigma}}$$



$$= \sigma$$

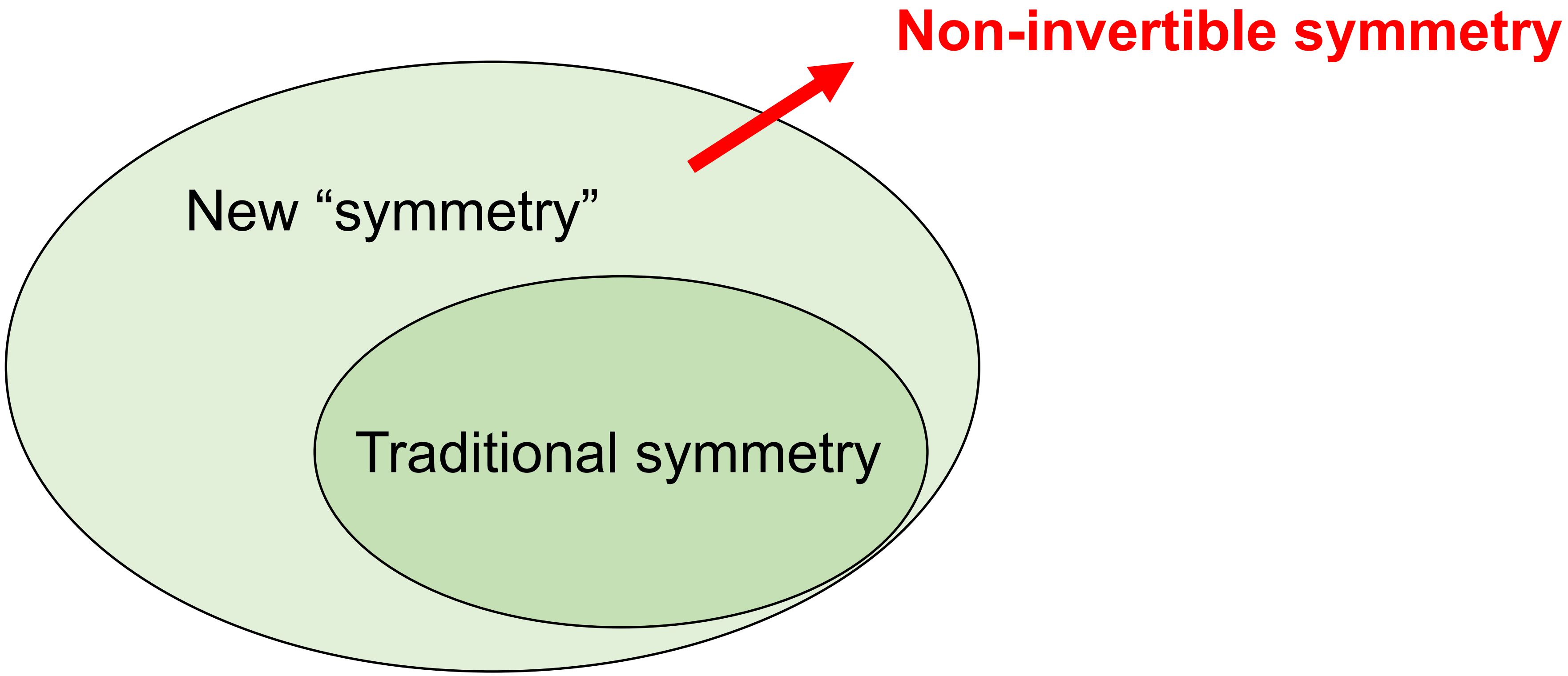
We can calculate arbitrary configuration of these defects.

Plan:

- Symmetry ⇒ topological defect
- Generalized symmetry
- Topological defects in 4d Z_2 lattice gauge theory — overview—
- Topological defects in 4d Z_2 lattice gauge theory — detail—
- Summary and discussion

Summary and discussion

Concept of symmetry is changing.



We find an example of non-invertible symmetry in 4 dimensions

[Koide, Nagoya, SY 21]

4-dimensional Z_2 lattice gauge theory

Duality [Wegner 71]

1-form Z_2 center symmetry

→ **Non-invertible symmetry**

Crossing relations and some expectation values are calculated.

This symmetry will not be only a special symmetry of a special theory, but it appears in many theories as KW duality in two dimensions.

Discussions

Applications?

Recently, a lot of examples of such non-invertible duality defects in 4-dimensional continuum quantum field theory have been found. [Choi, Cordova, Hsin, Lam, Shao 21], [Kaidi, Ohmori, Zheng 21]

They should be useful to analyze phase structures of QFTs.

AdS/CFT correspondence ⇒ string theory

The duality of N=4 SU(N) SYM is non-invertible.

⬇

The duality of type IIB string is non-invertible?

(cf anomaly for the duality of IIB [Debray, Dierigl, Heckman, Montero])