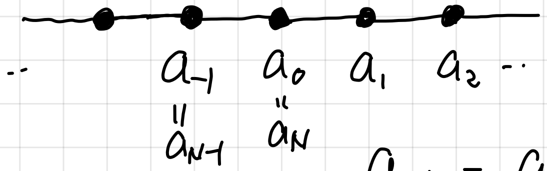
Non-invertible symmetry on the lattice

Plan

1. Classical statistical mechanics and operator formalism
2. 2d Ising model and its "symmetry"
3. 4d $\mathbb{Z}_2$ lattice gauge theory

1. Classical statistical mechanics and operator formalism

Eg. 1 dim Ising (sta. mech.)



$a_i = 0, 1$

$a_N = a_0$ (periodic boundary condition)

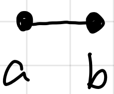
Partion function

$$Z = \sum_{\{a\}} \exp\left(K \sum_{i=0}^{N-1} (-1)^{a_i + a_{i+1}} \right)$$

K: parameter of the theory

$$= \sum_{\{a\}} \prod_{i=0}^{N-1} e^{K(-1)^{a_i + a_{i+1}}}$$

Transfer matrix $\longrightarrow$ Hilbert space, operators



$$= T_{ab} := e^{K(-1)^{a+b}}$$

2×2 matrix

$$T = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} \end{array}$$

Hilbert space $\mathcal{H} = \mathbb{C}^2$

$$T = \prod_{\{a\}} T_{a_0 a_1} T_{a_1 a_2} \cdots\cdots T_{a_{N-1} a_0}$$

summed

$\Rightarrow$ product of matrices

$$= \operatorname{tr} T^N$$

define H by $T = e^{-\frac{\beta}{N} H}$

$$\Rightarrow Z = \operatorname{tr} e^{-\beta H}$$

Partion function of quantum system

Two different pictures of a single system

Operator, Hilb. sp. vs Stat. mech.

Use this picture more in this talk.

Both are useful

※ To define QM, QFT, we have to take "continuum limit"

subtle! we do not consider here.

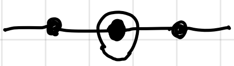
☆ Operator $\leftrightarrow$ defect

Eg.

a different bond is inserted here "defect"


a b $= D_{ab}$ $\Leftrightarrow$ operator on $\mathcal{H} = \mathbb{C}^2$

Eg.



$\underset{a}{\odot} = f(a)$ $\Rightarrow$ $\underset{a}{\odot} =$  a b $= f(a)\,\delta_{ab}$

※ A defect is NOT an excitation, but an arbitrary local change of the lattice

☆ Topological defect

A defect that commutes with T

$\sum_b$  $= \sum_b$  $\iff$ (In operator formalism $DT = TD$)

a b c   a b c

omitted later

No other defect is inserted here.

 =  = 

You can move a topological defect, unless it hits another defect

Eg.

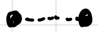
 $= \underbrace{(1-\delta_{ab})}_{\delta_{a,1-b}} = \eta_{ab}$ "spin flip"

a b

$$\left(\begin{aligned} &\sum_b \text{(a b c)} = \sum_b (1-\delta_{ab})\, e^{K(-1)^{b+c}} = e^{K(-1)^{a+c+1}} \\ &\sum_b \text{(a b c)} = \sum_b e^{K(-1)^{a+b}} (1-\delta_{bc}) = e^{K(-1)^{a+c+1}} \end{aligned} \right)$$

☆ Symmetry defect

- trivial defect : a topological defect

 $= \delta_{ab} =$ 

a b   a

※ c-num × δ_{ab} is also topological (c-number defect)

- Conjugate

      (hermitian conjugate as a matrix)

D $D^\dagger$

- Invertible defect

$$\Leftrightarrow \sum_b \underset{a}{\bullet} \xrightarrow{D} \underset{b}{\bullet} \xleftarrow{D^\dagger} \underset{c}{\bullet} = \underset{a}{\bullet} \cdots\cdots \underset{c}{\bullet}$$

trivial defect

($\Leftrightarrow$ D is unitary)

- Set of invertible topological defects forms a group $\tilde{G}$

= "symmetry"

group $G = \tilde{G} / \text{c-num}$

Eg.

$$\underset{a}{\bullet} \overset{\eta}{\text{———}} \underset{b}{\bullet} = \delta_{a,1-b} \qquad \eta^\dagger = \eta$$

$$\bullet\text{———}\bullet\text{———}\bullet = \bullet\cdots\cdots\bullet$$

$$G = \{1, \eta\} \cong \mathbb{Z}_2$$

☆ Non-invertible symmetry

= topological defects that is not invertible

Eg.

$$\underset{a}{\bullet}\text{———}\underset{b}{\bullet} = \delta_{ab} + \delta_{a,1-b} =: P_{ab}$$

$\rightsquigarrow$ topological

(a "condensation defect")

$$P^\dagger = P$$

satisfy

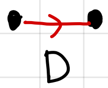
$$\bullet\text{———}\bullet\text{———}\bullet = 2\ \bullet\text{———}\bullet \qquad (\neq \bullet\cdots\cdots\bullet)$$

☆ Fusion

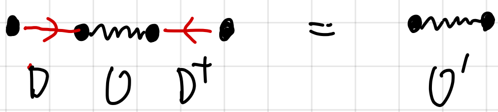
$i, j, \cdots$: labels of (invertible or non-invertible) topological defect

☆ Transformation

: a defect



: topological defect



" (Generalized) Ward-Takahashi identity "

If D is a symmetry defect,

$O \to O'$ is a representation of G

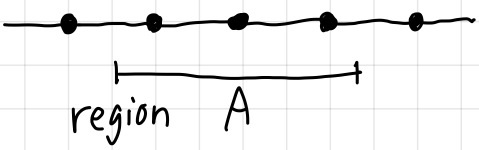
☆ Symmetry of the action and top. defect.

$$Z = \sum_{\{a\}} e^{-S(a)} \quad , \quad S(a) = \sum_{i=0}^{N-1} K (-1)^{a_i + a_{i+1}}$$

Symmetry : transformation $a \to a'$
s.t. $S(a) = S(a')$

$\Rightarrow$ inv. top. defect.

<u>Eg.</u> spin flip $\quad a_i \to 1 - a_i$



$$Z = \sum_{\{a\}} e^{-S(a)}$$

← Just change the letter

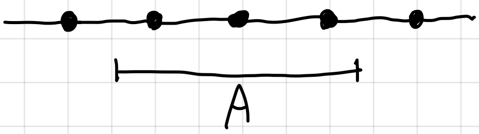
$$= \sum_{\{a'\}} e^{-S(a')}$$

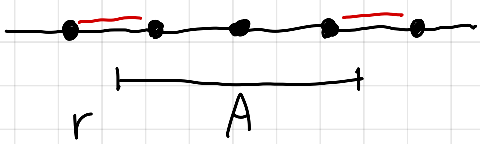
substitute

$$a'_i = \begin{cases} a_i & i \notin A \\ 1-a_i & i \in A \end{cases}$$

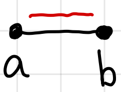
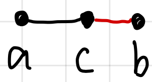
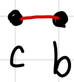
$$= \sum_{\{a\}} e^{-S_A(a)}$$

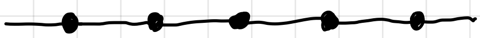
$S(a) \neq S_A(a)$ how ?

$S(a)$ 

$S_A(a)$ 

differences are only at ∂A

 $= e^{K(-1)^{a+1-b}}$ $=$  $\left(\right.$  $= \delta_{c,1-b} \left. \right)$



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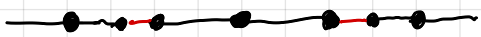
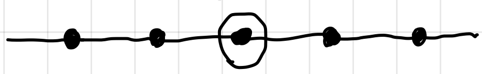


- Insertion of operator

$$Z \langle f(a_i) \rangle_{i \in A} = \sum_{\{a\}} e^{-S(a)} f(a_i) = \sum_{\{a\}} e^{-S_A(a)} f(1-a_i)$$

$f(a_i)$



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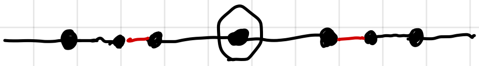
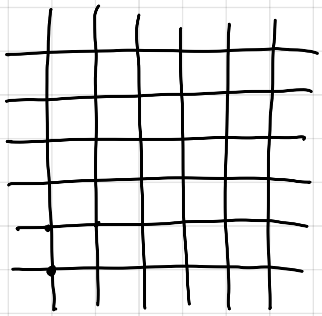


$f(1-a_i)$

2. 2d Ising and its symmetry

Eg.

2d Ising

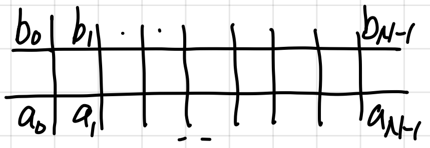


$a_i = 0, 1$

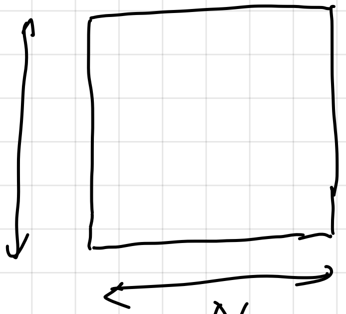
at each site i

$$Z = \sum_{\{a\}} \exp\left(K \sum_{\substack{\langle ij \rangle \\ \text{link}}} (-1)^{a_{\vec{i}} + a_{\vec{j}}} \right)$$

☆ Transfer matrix



$A = (a_0, \cdots, a_{N-1})$



periodic b.c.

$$T_{AB} = \exp\left(K \sum_{i=1}^{N} (-1)^{a_i + b_i} + \frac{1}{2} K \sum_{i=1}^{N} (-1)^{a_i + a_{i+1}} + \frac{1}{2} K \sum_{i=1}^{N} (-1)^{b_i + b_{i+1}} \right)$$

$\Downarrow$

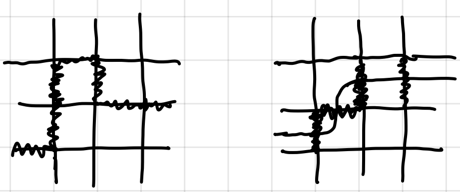
$Z = \mathrm{Tr}\, T^M$ Hilbert space $\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}$

☆ Defect

Point defect (codim 2)

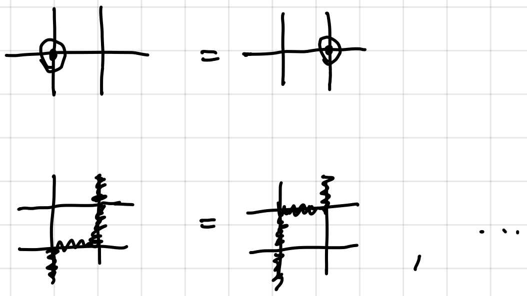
→ "local operator"

line defect



→ "line operator"

☆ Topological defect



☆ Symmetry defect

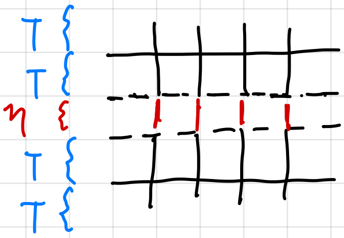
<u>Eg.</u> spin flip of Ising

Operator formalism

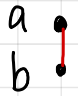
$$\eta = \sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \eta T = T \eta$$

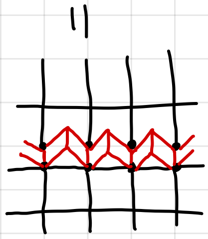
As a defect (realize $\cdots T\,T\,T\,\eta\,T\,T\,T\cdots$)

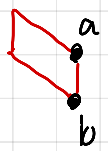


$= \exp\left(\frac{1}{2}K(-1)^{a+b}\right)$



$= \delta_{a,1-b} = (\sigma_x)_{ab}$

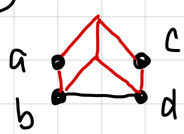




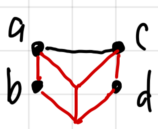
$= \delta_{a,1-b}$

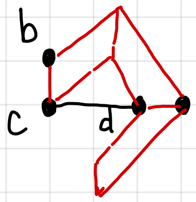
Not only commute with T, but also topological !

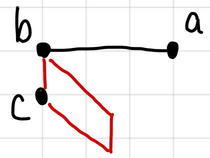
①

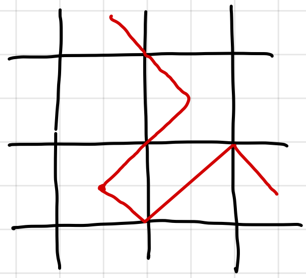


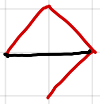
$=$

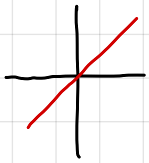
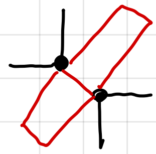


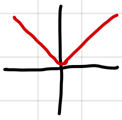
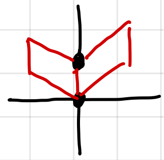
② $\sum_d$ 

= 



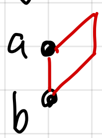
 = 

 ⇒ 

 ⇒ 

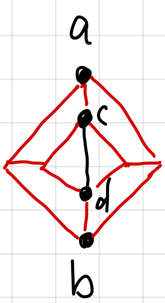
①, ② ⇒ η is topological

more general ansatz



$= r\, \delta_{a,1-b}$, ② ⇒ $r = \pm 1$

↑ c-num

③ 

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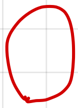
Invertible (stronger than topological)

 = 1

☆ Kramers - Wannier duality

$$Z = \sum_{\{b\}} \exp\Big(K \sum_{\ell} (-1)^{b_\ell}\Big) \prod_{p:\text{plaquette}} \delta^{\text{mod } 2}_{\ell_1+\ell_2+\ell_3+\ell_4,\,0}$$

$p = \langle \ell_1 \ell_2 \ell_3 \ell_4 \rangle$ ℓ_3, ℓ_2 $\square$ ℓ_4, ℓ_1

$b_\ell = 0, 1$ at each link

ℓ: link

Rewrite Z in two ways

(I)

This constraint

$\{b\} \underset{\sim}{\Leftrightarrow} \{a\}$ conf of spin at each site

s.t. ($\Leftarrow$) $b_{\langle ij \rangle} = a_i + a_j$ mod 2

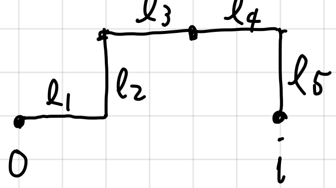
$\ell = \langle ij \rangle$

($\Rightarrow$) fix origin o, fix a_0

choose path $o \to i$ $\langle \ell_1 \ell_2 \cdots \ell_k \rangle$

$$a_i = a_0 + b_{\ell_1} + \cdots + b_{\ell_k}$$

(analogy : $\nabla \times E = 0$ $\Rightarrow$ $E = -\nabla\phi$, $\exists \phi$)



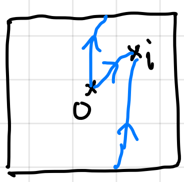
Independent of "continuous" deformation of path, due to the constraint.

To be precise,

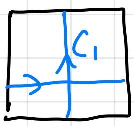
i) choice of $a_0 = 0, 1$

$\{b\} \underset{\sim}{\Rightarrow}$ two $\{a\}$

ii) Two paths from o to i may not be related by continuous deformation



take cure two non-trivial closed cycle



$\sum_{C_1} b = 0, 1$ Periodic Anti-periodic

$\sum_{C_2} b = 0, 1$ P A

There are four distinct sectors

$$PP,\ PA,\ AP,\ AA$$

$$\Rightarrow\ Z = \frac{1}{2} \sum_{\substack{PP, PA \\ AP, AA \\ \text{boundary} \\ \text{conditions}}} \sum_{\{a\}} \exp\left(K \sum_{\langle ij \rangle} (-1)^{a_i + a_j}\right) =: Z_{\mathrm{Ising}/\mathbb{Z}_2}(K)$$

Ising model, spin flip $\mathbb{Z}_2$ is topologically gauged.

(in string theory context "orbifold")

(II)

$$\delta^{\mathrm{mod}\,2}_{b_1+b_2+b_3+b_4,\,0} = \frac{1}{2} \sum_{c=0,1} (-1)^{c(b_1+b_2+b_3+b_4)}$$

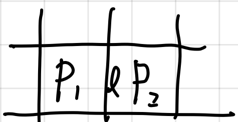
$$Z = \sum_{\{b\}} \sum_{\{c\}} \frac{1}{2^V} \exp\left(K \sum_{\substack{l \\ \text{links}}} (-1)^{b_l} + i\pi \sum_{\substack{p=\langle l_1 l_2 l_3 l_4 \rangle \\ \text{plaquette}}} c_p (b_{l_1} + b_{l_2} + b_{l_3} + b_{l_4})\right)$$

$c_p = 0, 1$ at each plaquette

$V = (\#\ \text{sites}) = (\#\ \text{plaquettes}) = \frac{1}{2}(\#\ \text{links})$

Summation of $\{b\}$ first

Each link belongs to two plaquettes



Pick up factors including b_l

$$\sum_{b_l=0,1} \exp\left(K(-1)^{b_l} + i\pi \underbrace{(c_{p_1} + c_{p_2})}_{=:c} b_l\right)$$

$$= e^{K} + (-1)^c e^{-K} = 2\cosh K (\tanh K)^c \underset{②}{=} \sqrt{\frac{2}{\sinh 2\tilde{K}}}\, e^{\tilde{K}(-1)^c}$$

$$\tilde{K} : \quad \tanh K = e^{-2\tilde{K}} \underset{①}{\Leftrightarrow} \mathrm{sh}\,2K\,\mathrm{sh}\,2\tilde{K} = 1$$

Calculation

① $$\frac{\mathrm{sh}\,K}{\mathrm{ch}\,K} = e^{-2\tilde{K}} \Rightarrow \frac{\mathrm{ch}\,K}{\mathrm{sh}\,K} = e^{2\tilde{K}}$$

$$\frac{\mathrm{ch}\,K}{\mathrm{sh}\,K} - \frac{\mathrm{sh}\,K}{\mathrm{ch}\,K} = e^{2\tilde{K}} - e^{-2\tilde{K}} = 2\,\mathrm{sh}\,2\tilde{K}$$

$$\text{l.h.s} = \frac{\mathrm{ch}^2K - \mathrm{sh}^2K}{\mathrm{sh}\,K\,\mathrm{ch}\,K} = \frac{1}{\frac{1}{2}\mathrm{sh}\,2K} \qquad \Rightarrow \mathrm{sh}\,2K\,\mathrm{sh}\,2\tilde{K} = 1$$

② $$\mathrm{sh}\,2K = 2\,\mathrm{ch}\,K\,\mathrm{sh}\,K = 2\,\mathrm{ch}^2K\,\underbrace{\mathrm{th}\,K}_{e^{-2\tilde{K}}}$$

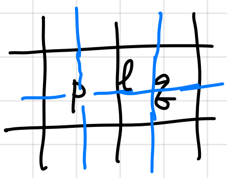
$$\Rightarrow \mathrm{ch}^2K = \frac{1}{2}\mathrm{sh}\,2K\,e^{2\tilde{K}}$$

$$\Rightarrow \mathrm{ch}\,K = \frac{1}{\sqrt{2}}\sqrt{\mathrm{sh}\,2K}\;e^{\tilde{K}}$$

$$2\,\mathrm{ch}\,K\,(\mathrm{th}\,K)^c = \sqrt{2\,\mathrm{sh}\,2K}\;e^{\overbrace{\tilde{K} - 2\tilde{K}c}^{= \tilde{K}(-1)^c}}$$

$$(-1)^c = 1 - 2c$$

$$Z = \sum_{\{c\}} \frac{1}{2^V} \prod_{\substack{l = \langle pq\rangle \\ \text{dual link}}} \sqrt{2\,\mathrm{sh}\,2K}\;e^{\tilde{K}(-1)^{c_p + c_q}}$$



$$(\#\text{link} = 2V)$$

$$= \frac{1}{(\mathrm{sh}\,2\tilde{K})^V} \underbrace{\sum_{\{c\}} e^{\tilde{K}\sum_{\langle pq\rangle}(-1)^{c_p + c_q}}} =: Z_{\text{Ising}}(\tilde{K})$$

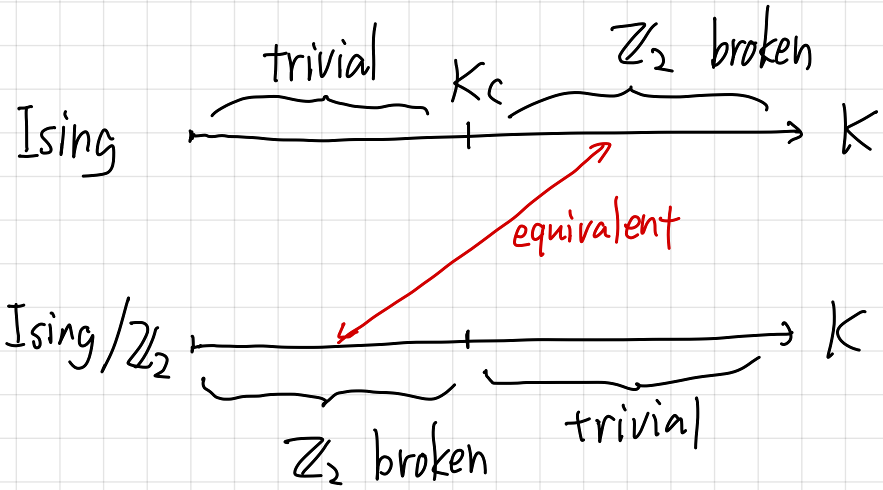
$$\boxed{\frac{1}{(\mathrm{sh}\,2K)^{1/2}} Z_{\mathrm{Ising}/\mathbb{Z}_2}(K) = \frac{1}{(\mathrm{sh}\,2\tilde{K})^{1/2}} Z_{\mathrm{Ising}}(\tilde{K})}$$

$$\boxed{\mathrm{sh}\,2\tilde{K}\;\mathrm{sh}\,2K = 1}$$

In particular

when $K = \tilde{K}$ $(= K_c)$ $\quad \left(\mathrm{sh}\,2K_c = \mathrm{sh}\,2\tilde{K}_c = 1 \right)$

$$Z_{\mathrm{Ising}/\mathbb{Z}_2}(K_c) = Z_{\mathrm{Ising}}(K_c)$$



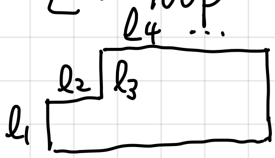
⊙ Fate of $\mathbb{Z}_2$ global symmetry

$$Z = \sum_{\{b\}} \exp\Big(K \sum_{\substack{\ell \\ \text{links}}} (-1)^{b_\ell} \Big) \prod_{\substack{p:\text{plaquette} \\ p=\langle \ell_1 \ell_2 \ell_3 \ell_4 \rangle}} \delta^{\bmod 2}_{\ell_1+\ell_2+\ell_3+\ell_4,\, 0}$$

- $\mathbb{Z}_2$ global symmetry

symmetry defect $\quad U_L := (-1)^{b_{\ell_1} + b_{\ell_2} + \cdots}$

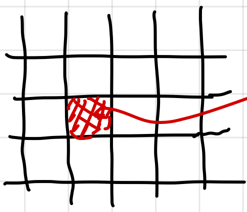
L : loop



U_L is topological since all plaquettes are trivial

$U_L^2 = 1 \quad \Rightarrow \mathbb{Z}_2$

- Charged defect

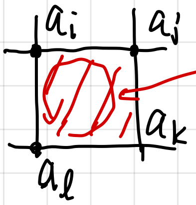


$\delta^{\bmod 2}_{\ell_1+\ell_2+\ell_3+\ell_4,\, 1}$

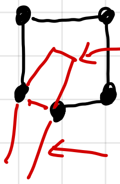
Ⓘ

$$Z = \frac{1}{2} \sum_{\text{B.C.}} \sum_{\{a\}} \exp\Big(K \sum_{\substack{\langle ij \rangle \\ \text{links}}} (-1)^{a_i + a_j} \Big)$$

- $U_L = (-1)^{a_{i_1} + a_{i_2} + \cdots} \qquad L = (\langle i_1 i_2 \rangle, \langle i_2 i_3 \rangle, \cdots)$

- 

anti-periodic boundary condition (monodromy, branch point, ..)

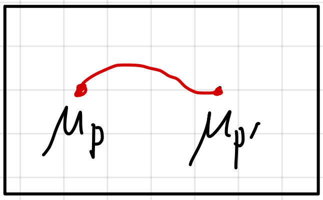
$=$ 

The end point of spin flip defect

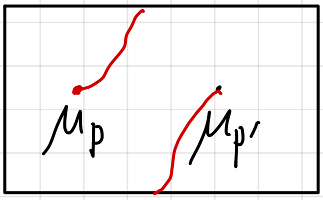
spin flip defect

μ_p

All conf of spin flip defect is summed.



$+$



$+$ ......

II

$$Z = \sum_{\{b\}}\sum_{\{c\}} \frac{1}{2^P}\exp\left(K\sum_\ell (-1)^{b_\ell} + i\pi \sum_{p=\langle \ell_1\ell_2\ell_3\ell_4\rangle} C_p(b_{\ell_1}+b_{\ell_2}+b_{\ell_3}+b_{\ell_4})\right)$$

$$U_L = (-1)^{b_{\ell_1}+b_{\ell_2}+\cdots}$$

$L \ni \ell$

$$\sum_{b_\ell=0,1}\exp\left(K(-1)^{b_\ell}+i\pi(C_{p_1}+C_{p_2})b_\ell\right)\quad(-1)^{b_\ell}$$

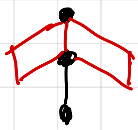
$$= e^K + (-1)^{C_{p_1}+C_{p_2}+1} e^{-K}$$

$$= \sqrt{\frac{2}{\sinh 2\tilde K}}\, e^{\tilde K(-1)^{C_{p_1}+C_{p_2}+1}}$$

anti-ferro interaction

dual lattice

$=$



$\mathbb{Z}_2$ symmetry defect of Ising model

μ_p

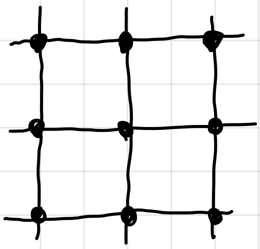
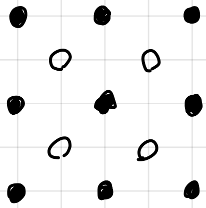
$$\delta^{\mathrm{mod}\,2}_{\ell_1+\ell_2+\ell_3+\ell_4,\,1} = \frac12\sum_c(-1)^{c(\ell_1+\ell_2+\ell_3+\ell_4+1)}$$

$$= \frac12\sum_c (-1)^{c(\ell_1+\ell_2+\ell_3+\ell_4)}(-1)^c$$

$\Rightarrow \quad \mu_p = (-1)^c$

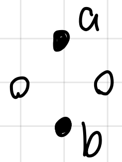
☆ KW defect [Aasen, Mong, Fendrey]

KW duality …. sites and dual sites are exchanged

 ⇒ 

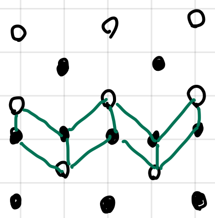
We do not put any degrees of freedom on ○

put weight for

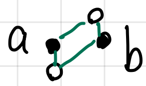


$= e^{K(-1)^{a+b}}$

Ansats :



○ ⟺ ● are exchanged across the KW defect.

building block 

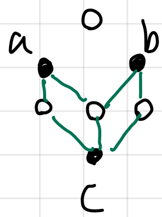
$= N(a, b)$

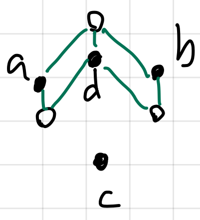
Assign weights for sites

● $=: u$

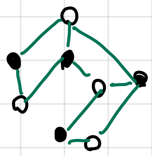
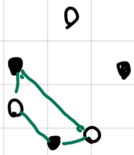
○ $=: v$

Require "defect commutation relation"



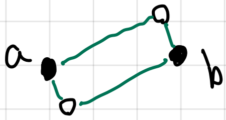
$=$


$$W(a,b)\,N(a,c)\,N(b,c)\,u^3 v^4 = \sum_d W(d,c)\,N(a,d)\,N(b,d)\,u^4 v^3$$

 $=$ 

⇓

Solution

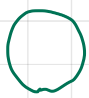


$= \frac{1}{\sqrt{2}}(-1)^{ab}$, $\sinh 2K = 1$

$\bullet = 1$

$\circ = \sqrt{2}$

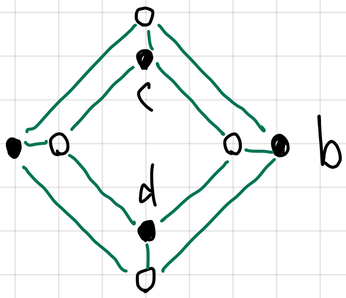
☆ Quantum dimension



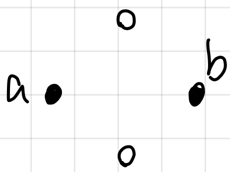
$= ?$

$$e^K - e^{-K} = 2$$
$$e^{2K} - 2e^K - 1 = 0$$
$$\begin{cases} e^K = \sqrt{1+\sqrt{2}} \\ e^{-K} = \sqrt{-1+\sqrt{2}} \end{cases}$$

$\sum_{c,d}$



$= \sqrt{2}$



Calculation

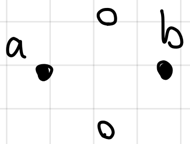
$$\sum_{c,d} N(a,c)N(a,d)N(b,c)N(b,d)\,W(c,d)\,u^4 v^4$$

$$= \sum_{c,d}\left(\frac{1}{\sqrt{2}}\right)^4 (-1)^{ac+ad+bc+bd}\;W(c,d)\;\sqrt{2}^4$$

$$= \sum_{c,d}(-1)^{(a+b)(c+d)}\;e^{K(-1)^{c+d}}$$

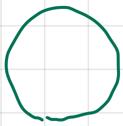
$K = K_c$ $\quad \mathrm{th}K = e^{-2K}$

$$= 2e^K + 2(-1)^{a+b}e^{-K} = \begin{cases} 4\,\mathrm{ch}K & a+b=0 \\ 4\,\mathrm{sh}K & a+b=1 \end{cases}$$



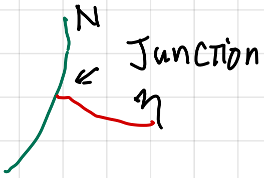
$$= u^2 v^2 W(a,b) = 2e^{K(-1)^{a+b}} = \begin{cases} 2e^{K} & a+b=0 \\ 2e^{-K} & a+b=1 \end{cases}$$

$$\begin{cases} a+b=0 \Rightarrow \dfrac{4\,\mathrm{ch}K}{2e^{K}} = 1+e^{-2K} = \sqrt{2} \\[2ex] a+b=-1 \Rightarrow \dfrac{4\,\mathrm{sh}K}{2e^{-K}} = e^{2K}-1 = \sqrt{2} \end{cases}$$



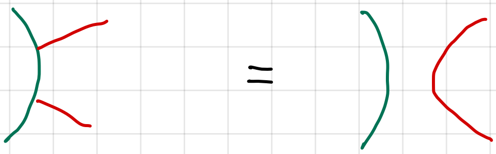
$= \sqrt{2}$ Non-invertible !

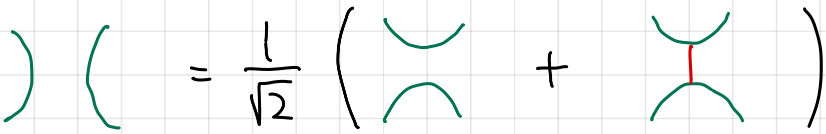
☆ Junction and crossing relation

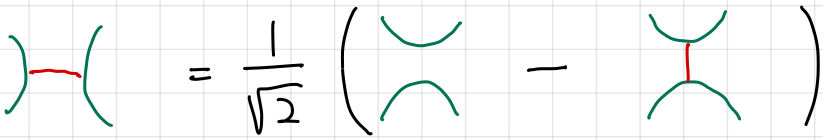


weight at this point

Crossing relations







$\rightsquigarrow$ structure of "fusion category"

3. 4D $\mathbb{Z}_2$ lattice gauge theory

4D analog of 2D Ising

4D cubic lattice

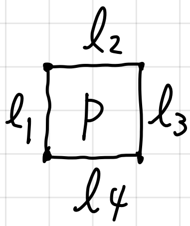
Put $a_\ell = 0, 1$ for each link ℓ

$$Z = \frac{1}{2^V} \sum_{\{a\}} \exp\left(K \sum_{p} (-1)^{a_{\ell_1} + a_{\ell_2} + a_{\ell_3} + a_{\ell_4}}\right)$$

p : plaquette

$p = \langle \ell_1 \ell_2 \ell_3 \ell_4 \rangle$

(V : # sites)



☆ Gauge symmetry

parameter $\lambda_i = 0, 1$ i : sites

$$a_{\langle ij \rangle} \to a_{\langle ij \rangle} + \lambda_i - \lambda_j \mod 2$$

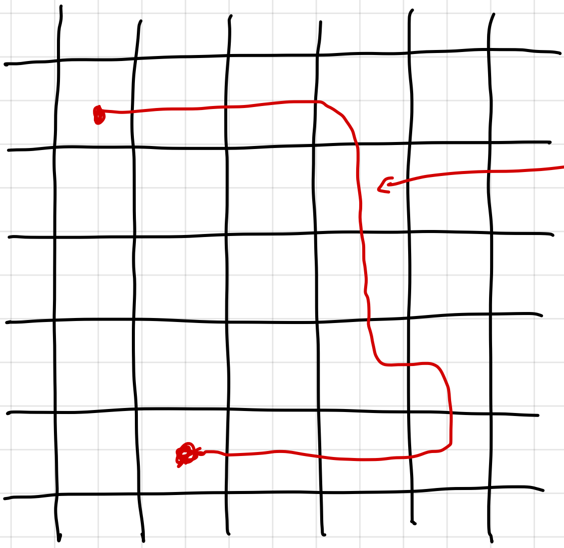
plaquette is invariant

※ $\mathbb{Z}_2$ lattice gauge theory $\neq$ (topological) $\mathbb{Z}_2$ gauge theory

☆ Center symmetry (analog of spin flip symmetry of 2D Ising)

codim 2 topological defect with $\mathbb{Z}_2$ structure

1-form symmetry



M : 3 dim submfd with ∂

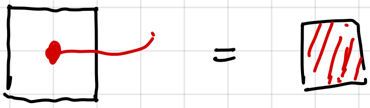
transformation

$$a'_\ell = \begin{cases} 1 - a_\ell & \text{if } \ell \text{ cross } M \\ a_\ell & \text{otherwise} \end{cases}$$

$$Z = \sum_{\{a\}} e^{-S(a)} = \sum_{\{a'\}} e^{-S(a')} = \sum_{\{a\}} e^{-S_M(a)}$$

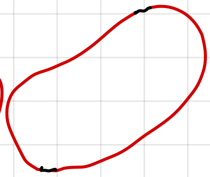
The difference is localized on $\Sigma = \partial M$

$\uparrow$ 2 dim



$$= e^{-K(-1)^{a_{\ell_1}+a_{\ell_2}+a_{\ell_3}+a_{\ell_4}}}$$

$\Sigma = \partial M$



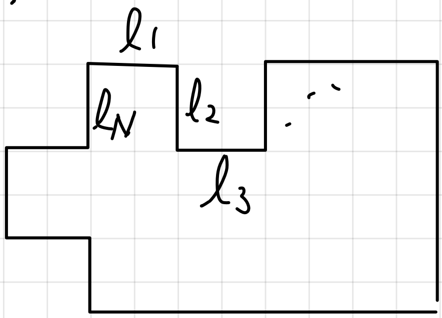
$\eta(\Sigma)$ $= 1$

☆ Wilson loop

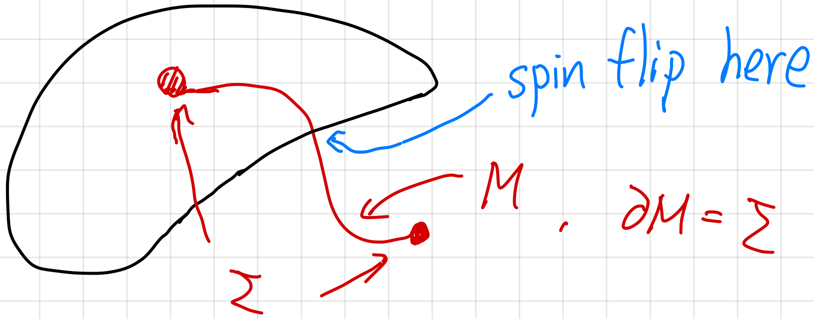
L : a loop made by connecting links

$$W_L = (-1)^{a_{\ell_1}+a_{\ell_2}+\cdots+a_{\ell_N}}$$

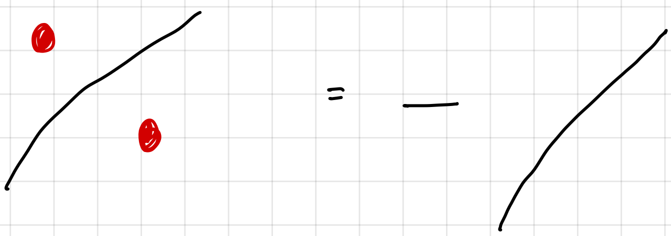
$L=(\ell_1,\ell_2,\ell_3,\cdots,\ell_N)$



Wilson loop is charged under the center symmetry



, $\partial M = \Sigma$



☆ Confinement, deconfinement

K : small ($\Leftrightarrow$ strong coupling)

L : large $\quad \langle W_L \rangle \sim e^{-T(\text{Area})}$: Area law

$T \neq 0$

confinement

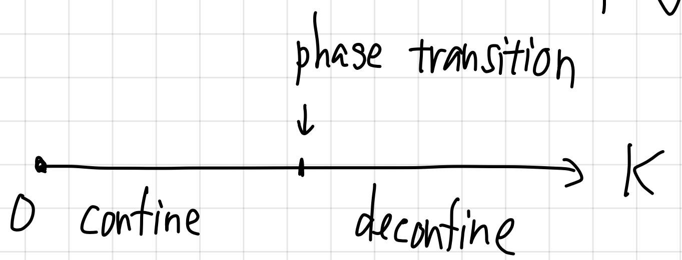
center sym not broken

trivial phase (SRE)

K : large ($\Leftrightarrow$ weak coupling)

$$\langle W_L \rangle \sim e^{-m(\text{perimeter})} \quad : \text{perimeter law}$$

deconfinement
cent. sym. broken
topologically ordered phase



☆ KWW duality (Wegner)

Derived same way as 2D Ising

T : 4d $\mathbb{Z}_2$ lattice gauge th.

$$\frac{1}{(\mathrm{sh}2K)^{3V/2}} Z_{T/\mathbb{Z}_2^{[1]}}(K) = \frac{1}{(\mathrm{sh}2\tilde{K})^{3V/2}} Z_T(\tilde{K})$$

V : # vertices

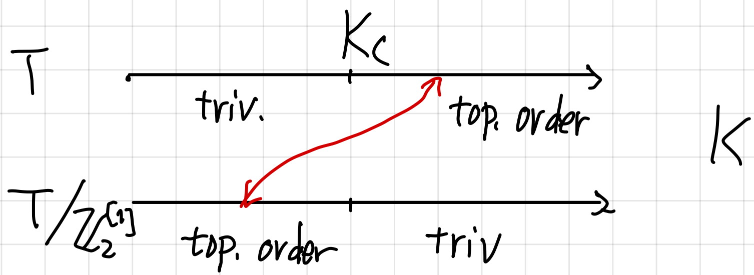
$$\mathrm{sh}2K\, \mathrm{sh}2\tilde{K} = 1$$

Self-dual point $\quad \mathrm{sh}2K_c = 1$

Most likely, the phase transition occurs at $K = K_c$

$$\Rightarrow \quad Z_{T/\mathbb{Z}_2^{[1]}}(K_c) = Z_T(K_c)$$

centersymmetry is gauged.



◎ Weak coupling limit

$$Z = \frac{1}{2^V} \sum_{\{a\}} \exp\left(K \sum_p S_p(a)\right)$$

$$S_p(a) = (-1)^{a_{l_1} + a_{l_2} + a_{l_3} + a_{l_4}}$$

$$p = \langle l_1 l_2 l_3 l_4 \rangle$$

$K \to \infty$, $S_p(a) = -1$, $\exists p$ is highly surpressed.

$\Rightarrow$ Only "flat" gauge configurations contribute

$$\forall p \quad S_p(a) = 1$$

$\Rightarrow$ topological $\mathbb{Z}_2$ gauge theory

$$\begin{cases} \langle W_L \rangle = 1 & \text{if } L \text{ can be shrinked} \\ \langle W_L \rangle = 0 & L : \text{non-trivial cycle} \end{cases}$$

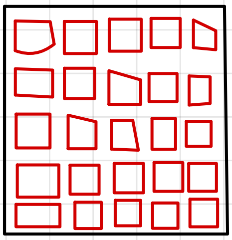
◎ Strong coupling limit

$$\langle W_L \rangle = \frac{1}{Z}\,\frac{1}{2^V}\sum_{\{a\}}\Big(\underset{\times (-1)^{a_{l_1}+a_{l_2}+\cdots+a_{l_N}}}{1} + K\sum_{p} S_p(a) + \frac{1}{2}\Big(K\sum_{p} S_p(a)\Big)^2 + \cdots \Big)$$

$$\sum_{\{a\}} (-1)^{a_{l_1}+\cdots} \propto \sum_{a_l=0,1} (-1)^{a_{l_1}} = 0$$

$(-1)^{a_{l_1}}$ must be provided from here

If a single link variable $(-1)^{a_l}$ appears, the summation vanishes



$$\propto K^{\overset{\text{Area}}{A}} = e^{-A|\log K|}$$

$$\left(\begin{array}{l} K \ll 1 \\ \log K < 0 \end{array} \right)$$

area law

= confinement.