QFT II. Homework Problem Set 5. (11/18/2016)

Due 12/9/2016

I. Let us evaluate 1-loop contribution to the propagator

$$i\Pi_1(p^2) := \frac{1}{2}(ig\tilde{\mu}^{\epsilon/2})^2 i^2 \int_q \frac{1}{(q^2-m^2)((p-q)^2-m^2)},$$

where

$$\int_q := \int \frac{d^d q}{(2\pi)^d}.$$

$i\varepsilon$ prescription is implicitly imposed as $m^2 = \mathrm{Re}\, m^2 - i\varepsilon$.

(1) Show the Feynman parameter formula

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{((1-x)A + xB)^2}.$$

(2) Apply the above formula with $A = q^2 - m^2$, $B = (p-q)^2 - m^2$ and complete square as

$$(1-x)A + xB = (q-a)^2 - D.$$

Find q-independent vector a and scalar D.

Let us change the integration valuable from q to $\ell = q - a$. Then deform the integration path of ℓ^0 from $-\infty \to \infty$ to $-i\infty \to i\infty$ ("Wick rotation"). Let $i\ell^d := \ell^0$ the integral is written as

$$i\Pi_1(p^2) := \frac{1}{2}(ig\tilde{\mu}^{\epsilon/2})^2 i^2 i \int_0^1 dx \int_{\hat{\ell}} \frac{1}{(\hat{\ell}^2 + D)^2},$$

where $\hat{\ell} := (\ell^1, \ldots, \ell^d)$ and $\hat{\ell}^2$ is the Euclidean length square $\hat{\ell}^2 = \sum_{i=1}^d (\ell^i)^2$.

(3) Derive the formula

$$\int_{\hat{\ell}} \frac{1}{(\hat{\ell}^2 + D)^n} = \frac{\Gamma(n - d/2)}{(4\pi)^{d/2}\Gamma(n)} \frac{1}{D^{n-d/2}}.$$

Hint: derive and use the formula

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty dt\, t^{n-1} e^{-At}.$$

(4) Use this formula and show

$$\Pi_1(p^2) = \frac{1}{2}\alpha\Gamma(-1+\epsilon/2) \int_0^1 dx D \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2},$$

where $\epsilon = 6 - d$. (continued overleaf)

(5) Expand this formula around $\epsilon = 0$ and obtain

$$\Pi_1(p^2) = \alpha\left[\left(\frac{1}{\epsilon} + \frac{1}{2}\right)\left(\frac{1}{6}p^2 - m^2\right) - \frac{1}{2}\int_0^1 dx D \ln\left(\frac{\mu^2}{D}\right)\right] + O(\epsilon),$$

where $\mu = \sqrt{4\pi}e^{-\gamma/2}\tilde{\mu}$, and γ is the Euler constant.

Some useful formulas.

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma + \sum_{k=1}^{n}\frac{1}{k}\right] + O(\epsilon), \quad (n > 0,\ \text{integer}).$$

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln A + O(\epsilon)$$

II.

Next let us evaluate the following integral which appears in the 1-loop diagram for the 3 point vertex.

$$i\boldsymbol{V}_{3,1} = (ig\tilde{\mu}^{\epsilon/2})^3 i^3 \int_q \frac{1}{((q - p_1)^2 - m^2)((q + p_2)^2 - m^2)(q^2 - m^2)}$$

The useful Feynman parameter formula here is

$$\frac{1}{ABC} = \int dF_3 \frac{1}{(x_1 A + x_2 B + x_3 C)^3},$$

$$\int dF_3 := 2\int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1).$$

(6) Apply the above formula with $A = (q - p_1)^2 - m^2, B = (q + p_2)^2 - m^2, C = q^2 - m^2$ and complete square as

$$x_1 A + x_2 B + x_3 C = (q - a)^2 - D$$

Find q-independent vector a and scalar D.

(7) Use the Wick rotation and show that

$$\boldsymbol{V}_{3,1}/(g\tilde{\mu}^{\epsilon/2}) = \frac{1}{2}\alpha\Gamma\left(\frac{\epsilon}{2}\right)\int dF_3\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2}.$$

(8) Expand the above result around $\epsilon = 0$ and find

$$\boldsymbol{V}_{3,1}/(g\tilde{\mu}^{\epsilon/2}) = \frac{1}{2}\alpha\left(\frac{2}{\epsilon} + \int dF_3 \ln\frac{\mu^2}{D} + O(\epsilon)\right).$$

It will be useful to notice that

$$\int dF_3 1 = 1.$$