1 4-dimensional Weyl spinor

- Left moving $\psi_\alpha,\ \alpha = 1, 2$
- Right moving $\psi_{\dot\alpha},\ \dot\alpha = 1, 2$

They are related by the complex conjugation. The indices are raised or lowerd by the ϵ tensor as

$$\psi^\alpha := \epsilon^{\alpha\beta}\psi_\beta, \qquad \bar\psi^{\dot\alpha} := \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}. \tag{1.1}$$

Then

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \qquad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}, \tag{1.2}$$

where

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \qquad \epsilon^{12} = +1, \quad \epsilon_{12} = -1, \tag{1.3}$$

and similar for one with dotted indices.

Inner product

$$\psi\chi := \psi^\alpha\chi_\alpha, \quad \bar\psi\bar\chi := \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}. \tag{1.4}$$

Sigma matrices $\sigma^\mu_{\alpha\dot\beta}$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

We also define $\bar\sigma^{\mu\ \dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$, or $\bar\sigma^0 = \sigma^0,\ \bar\sigma^i = -\sigma^i,\ (i = 1, 2, 3)$. Then they satisfy

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = -2\eta^{\mu\nu}, \tag{1.6}$$

$$\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = -2\eta^{\mu\nu}. \tag{1.7}$$

Lorentz generators

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\nu), \qquad \bar\sigma^{\mu\nu} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\nu) \tag{1.8}$$

Spinor bilinear

$$\psi\sigma^\mu\bar\chi := \psi^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\chi^{\dot\beta}, \qquad \bar\psi\bar\sigma^\mu\chi := \bar\psi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\chi_\beta. \tag{1.9}$$

Hermitian conjugations

$$(\psi\chi)^\dagger = \bar\psi\bar\chi, \qquad (\bar\psi\bar\chi)^\dagger = \psi\chi, \qquad (\psi\sigma^\mu\bar\chi)^\dagger = \chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi, \qquad (\bar\psi\bar\sigma^\mu\chi)^\dagger = \bar\chi\bar\sigma^\mu\psi = -\psi\sigma^\mu\bar\chi, \tag{1.10}$$

Some useful formulae

$$\psi\chi = \chi\psi, \qquad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}, \qquad \psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi, \qquad \psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi, \qquad \bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\chi} = -\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\psi}, \quad (1.11)$$

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \qquad \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \qquad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \tag{1.12}$$

$$(\theta\chi)(\theta\psi) = -\frac{1}{2}(\theta\theta)(\chi\psi), \qquad (\bar{\theta}\bar{\chi})(\bar{\theta}\bar{\psi}) = -\frac{1}{2}(\bar{\theta}\bar{\theta})(\bar{\chi}\bar{\psi}), \qquad \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}, \tag{1.13}$$

$$\chi\sigma^{\mu\nu}\theta\psi\sigma^{\rho\sigma}\theta = \frac{1}{2}\theta\theta\chi\sigma^{\mu\nu}\sigma^{\rho\sigma}\psi, \qquad -\epsilon^{\alpha\beta}(\sigma^{\mu\nu}\theta)_\alpha(\sigma^{\mu\nu}\theta)_\beta = \frac{1}{4}\theta\theta(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} + i\epsilon^{\mu\nu\rho\sigma}), \tag{1.14}$$

$$\sigma^\mu\bar{\sigma}^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}, \qquad \bar{\sigma}^\mu\sigma^\nu = -\eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu}, \qquad \mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] = -2\eta^{\mu\nu}, \tag{1.15}$$

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}, \qquad (\epsilon^{0123} = +1). \tag{1.16}$$

2 Superspace

Superspace coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

A superfiled $F(x, \theta, \bar{\theta})$

Differential operators

$$Q_\alpha := \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \overline{Q}_{\dot{\alpha}} := -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \tag{2.1}$$

$$\mathcal{D}_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \overline{\mathcal{D}}_{\dot{\alpha}} := -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{2.2}$$

Anti-commutation relations

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \qquad \{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \tag{2.3}$$

$$\{Q_\alpha, Q_\beta\} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{D}}_{\dot{\beta}}\} = 0, \tag{2.4}$$

$$\{Q_\alpha, \mathcal{D}_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \mathcal{D}_\beta\} = \{Q_\alpha, \overline{\mathcal{D}}_{\dot{\beta}}\} = \{\overline{Q}_{\dot{\alpha}}, \overline{\mathcal{D}}_{\dot{\beta}}\} = 0. \tag{2.5}$$

SUSY transformation with parameters $\xi^\alpha, \bar{\xi}^{\dot{\alpha}}$

$$\delta_\xi F(x, \theta, \bar{\theta}) := (\xi Q + \bar{\xi}\overline{Q})F(x, \theta, \bar{\theta}). \tag{2.6}$$

3 Chiral superfield

A chiral superfield $\Phi(x,\theta,\bar{\theta})$ satisfies

$$\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0. \tag{3.1}$$

Let us introduce $y^\mu := x^\mu + i\theta\sigma^\mu\bar{\theta}$. By the superspace coordinates $(y,\theta,\bar{\theta})$, the differential operators are written as

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha}, \qquad \overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} = \overline{\mathcal{D}}_{\dot{\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \tag{3.2}$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \qquad \overline{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \tag{3.3}$$

It is expanded as

$$\Phi(x,\theta,\bar{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{3.4}$$

$$= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x). \tag{3.5}$$

The complex conjugate of a chiral superfield satisfies $\mathcal{D}_\alpha\overline{\Phi} = 0$ and it is expanded as

$$\overline{\Phi}(x,\theta,\bar{\theta}) = \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \bar{\theta}\bar{\theta}\overline{F}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{\phi}(x). \tag{3.6}$$

SUSY transformation

$$\delta_\xi\phi = \sqrt{2}\xi\psi, \qquad \delta_\xi\bar{\phi} = \sqrt{2}\bar{\xi}\bar{\psi}, \tag{3.7}$$

$$\delta_\xi\psi_\alpha = \sqrt{2}\xi_\alpha F + \sqrt{2}i(\sigma^\mu\bar{\xi})_\alpha\partial_\mu\phi, \qquad \delta_\xi\bar{\psi}^{\dot{\alpha}} = \sqrt{2}\bar{\xi}^{\dot{\alpha}}\overline{F} + \sqrt{2}i(\bar{\sigma}^\mu\xi)^{\dot{\alpha}}\partial_\mu\bar{\phi}, \tag{3.8}$$

$$\delta_\xi F = \sqrt{2}i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi, \qquad \delta_\xi F = \sqrt{2}i\xi\bar{\sigma}^\mu\partial_\mu\bar{\psi}. \tag{3.9}$$

3.1 Super potential term

Let us consider n chiral superfields Φ^i, $i = 1,\dots,n$ and their complex conjugates $\overline{\Phi}^{\bar{\imath}}$, $\bar{\imath} = 1,\dots,n$. A possible SUSY invariant term is a super potential term.

$$\mathcal{L}_W = W(\Phi)|_{\theta\theta} + (\text{c.c.}), \tag{3.10}$$

where $W(\phi)$ is a holomorphic function of $\phi^1,\dots,\phi^n$, called "super potential." This super potential term is expanded as

$$\mathcal{L}_W = F^i\partial_i W - \frac{1}{2}\partial_i\partial_j W\psi^i\psi^j + (\text{c.c.}). \tag{3.11}$$

3.2 Kähler potential term

Another possible SUSY invariant term is the Kähler potential term.

$$\mathcal{L}_K = K(\Phi, \overline{\Phi})|_{\theta\theta\bar{\theta}\bar{\theta}}, \tag{3.12}$$

where $K(\phi, \bar{\phi})$ is a real function, called "Kähler potential." It is expanded as

$$\begin{aligned}\mathcal{L}_K =& g_{i\bar{j}}\left(-\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu\bar{\psi}^{\bar{j}} - \frac{i}{2}\bar{\psi}^{\bar{j}}\bar{\sigma}^\mu\partial_\mu\psi^i + F^i\overline{F}^{\bar{j}}\right) \\ &- \frac{i}{2}K_{ij\bar{k}}\bar{\psi}^{\bar{k}}\bar{\sigma}^\mu\psi^i\partial_\mu\phi^j - \frac{i}{2}K_{\bar{\imath}\bar{j}k}\psi^k\sigma^\mu\bar{\psi}^{\bar{\imath}}\partial_\mu\bar{\phi}^{\bar{j}} \\ &- \frac{1}{2}K_{ij\bar{k}}\psi^i\psi^j\overline{F}^{\bar{k}} - \frac{1}{2}K_{\bar{\imath}\bar{j}k}\bar{\psi}^{\bar{\imath}}\psi^{\bar{j}}F^k + \frac{1}{4}K_{ij\bar{k}\bar{\ell}}\psi^i\psi^j\bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{\ell}},\end{aligned} \tag{3.13}$$

where

$$K_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q} := \frac{\partial}{\partial\phi^{i_1}}\cdots\frac{\partial}{\partial\phi^{i_p}}\frac{\partial}{\partial\bar{\phi}^{\bar{j}_1}}\cdots\frac{\partial}{\partial\phi^{\bar{j}_q}}K(\phi, \bar{\phi}), \qquad g_{i\bar{j}} := K_{i\bar{j}}. \tag{3.14}$$

4 Vector superfield

A super field $V(x, \theta, \bar{\theta})$ is called "vector super field" if it is real $V^\dagger = V$. A vector super field is used to describe a gauge theory. For a $U(1)$ gauge theory the gauge parameter is a chiral super field $\Lambda(x, \theta, \bar{\theta})$, $\overline{\mathcal{D}}_{\dot{\alpha}}\Lambda = 0$, and transformation law is

$$V \to V' = V + \Lambda + \overline{\Lambda}. \tag{4.1}$$

The following "field strength" is gauge invariant.

$$W_\alpha := -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha V. \tag{4.2}$$

This W_α is a chiral super field, i.e. $\overline{\mathcal{D}}_{\dot{\alpha}}W_\alpha = 0$. It is also checked that it satisfied "the reality condition" $\mathcal{D}W = \overline{\mathcal{D}W}$.

It is convenient to choose "Wess-Zumino gauge", in which V is expanded as

$$V(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \tag{4.3}$$

It is convenient to rewrite it as

$$V = -\theta\sigma^\mu\bar{\theta}v_\mu(y) + i\theta\theta\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D(y) - i\partial^\mu v_\mu(y)). \tag{4.4}$$

W_α is calculated as

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - i(\sigma^{\mu\nu}\theta)_\alpha v_{\mu\nu}(y) + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(y))_\alpha, \tag{4.5}$$

and the bilinear of W_α

$$\frac{1}{4}WW = -\frac{1}{4}\lambda(y)\lambda(y) - \frac{i}{2}\theta\lambda(y)D(y) - \frac{1}{2}\lambda\sigma^{\mu\nu}\theta v_{\mu\nu}(y) + \theta\theta\left(-\frac{i}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{4}D^2 - \frac{1}{8}v_{\mu\nu}v^{\mu\nu} - \frac{i}{8}v_{\mu\nu}\tilde{v}^{\mu\nu}\right), \tag{4.6}$$

where $\tilde{v}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_{\rho\sigma}$. A gauge invariant Lagrangian can be written, with a complex constant $\tau_0 = \tau_{01} + i\tau_{02}$

$$\mathcal{L}_V = \frac{-i}{8\pi}\tau_0 WW|_{\theta\theta} + (\text{c.c.}) \tag{4.7}$$

$$= \frac{\tau_{02}}{2\pi}\left(-\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - \frac{i}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{i}{2}\bar{\lambda}\bar{\sigma}\partial_\mu\lambda + \frac{1}{2}D^2\right) - \frac{\tau_{01}}{8\pi}v_{\mu\nu}\tilde{v}^{\mu\nu}. \tag{4.8}$$

5 $\mathcal{N} = 2$ Lagrangian

An $\mathcal{N} = 2$ vector multiplet contains an $\mathcal{N} = 1$ vector multiplet and an $\mathcal{N} = 1$ chiral multiplet.

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - i(\sigma^{\mu\nu}\theta)_\alpha v_{\mu\nu}(y) + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(y))_\alpha, \tag{5.1}$$

$$A = a(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{5.2}$$

Since we do not have potential term for a, we do not have superpotential for A. Thus the generic Lagrangian which preserves $\mathcal{N} = 1$ SUSY can be written as

$$\mathcal{L} = K(A, \overline{A})|_{\theta\theta\bar{\theta}\bar{\theta}} + \frac{-i}{8\pi}\tau(A)WW|_{\theta\theta} + (\text{c.c.}), \tag{5.3}$$

where $\tau(a) = \tau_1(a) + i\tau_2(a)$ is a holomorphic function of a. Let us see the kinetic terms of the fermions

$$\mathcal{L} = g_{a\bar{a}}\left(-\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right) + \frac{\tau_2}{2\pi}\left(-\frac{i}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{i}{2}\bar{\lambda}\bar{\sigma}\partial_\mu\lambda\right) + \cdots. \tag{5.4}$$

In order to have $\mathcal{N} = 2$ SUSY, the kinetic term for ψ and λ must be the same. Thus

$$g_{a\bar{a}} = \frac{\tau_2}{2\pi} = \frac{1}{4\pi i}(\tau(a) - \overline{\tau(a)}). \tag{5.5}$$

In other words, there is a holomorphic function $\mathcal{F}(a)$ which satisfies

$$\mathcal{F}''(a) = \tau(a), \qquad K(a, \bar{a}) = \frac{1}{4\pi i}(\bar{a}\,\mathcal{F}'(a) - a\,\overline{\mathcal{F}'(a)}). \tag{5.6}$$

If we obtain this holomorphic function $\mathcal{F}(a)$, we can completely determine the action.