The index of lattice Dirac operators and K-theory

OSAKA UNIVERSITY
Live Locally, Grow Globally

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Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), "The index of lattice Dirac operators and K-theory," **arXiv:2407.17708**

What is the index of Dirac operators?

$$D\psi = 0 \quad D := \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

we consider
U(1) or SU(N) group

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) = \boldsymbol{E}\cdot\boldsymbol{B}$$

Topological charge

#sol with + chirality

#sol with - chirality

Very important both in physics and mathematics to understand gauge field topology, which is nonperturbative.

Physicist-friendly index project in continuum

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly APS index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two APS index [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]

Q. How physicist-friendly?
A. We do not need to take care of chiral symmetry and boundary conditions in our formulation.

This work = the first lattice version.

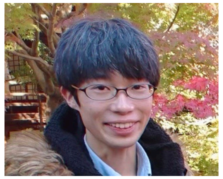
We mathematically "reformulate" the standard Atiyah-Singer index on an even-dimensional flat periodic lattice(, whose continuum limit is the Dirac index on a torus).

In our formulation

- No chiral symmetry is needed : massive Wilson Dirac operator is enough to consider.
- K theory is used to show equality to the continuum Dirac index.
- Wider application than the overlap Dirac operator.
- Mathematically very nontrivial (main dish for mathematicians).

Phys-Math collaborators

Physicists



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3. K-theory

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7. Summary and discussion

Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:
If $\gamma_5 D + D\gamma_5 = 0,$ we cannot avoid fermion doubling.

since the lattice discretization

$p_\mu \to \frac{1}{a}\sin(p_\mu a)$ gives unphysical poles $p_\mu = 0, \quad \frac{\pi}{a}$

a :lattice spacing

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

can avoid NN theorem.
But no concrete form was found in ~20 years.

Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \quad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = a D_{ov}\gamma_5 D_{ov}$

and the action

$$S = \sum_x \bar{q}(x) D_{ov} q(x)$$

is invariant under the modified chiral rotation:

$$q \to e^{i\alpha\gamma_5(1 - aD_{ov})} q, \quad \bar{q} \to \bar{q} e^{i\alpha\gamma_5}.$$

[Luescher 1998]

Anomaly and index of the overlap Dirac operator

Moreover, it reproduces the anomaly.

$$q \to e^{i\alpha\gamma_5(1-aD_{ov})}q, \quad \bar{q} \to \bar{q}e^{i\alpha\gamma_5}.$$

$$Dq\bar{q} \to \exp\left[2i\alpha\mathrm{Tr}(\gamma_5 + \gamma_5(1-aD_{ov}))/2\right] Dq\bar{q}$$

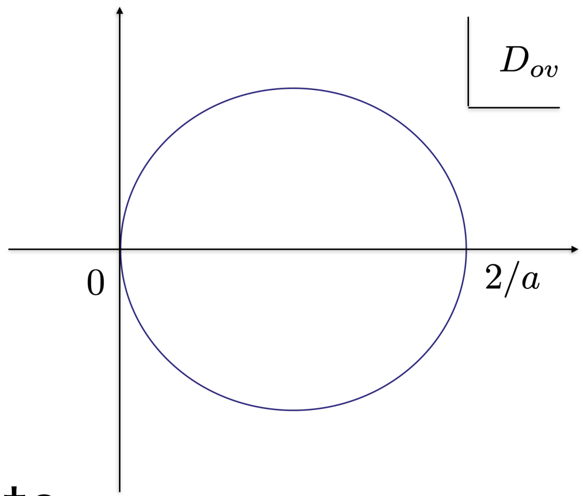
and the index is well-defined:

$$\mathrm{Ind}D_{ov} = \mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right)$$

[Hasenfratz et al. 1998]

The overlap Dirac operator index

a :lattice spacing



Overlap Dirac spectrum lies on a circle with radius 1/a

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)\psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real 2/a (doubler poles) do not contribute.

$$\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\text{zero-modes}} \gamma_5 \quad = n_+ - n_-$$

But D_{ov} is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5 (D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

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$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

What is this ???

η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}} \mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

The Wilson Dirac operator and K-theory

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

In this talk, we try to show a deeper mathematical meaning of the right-hand side of the equality,

and try to convince you that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…]

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What is fiber bundle?

A united manifold of spacetime (= base manifold) and field (fiber)

$$\phi(x) \;\rightarrow (x, \phi) \in X \times F$$

Spacetime
= base space

Field space
= fiber space

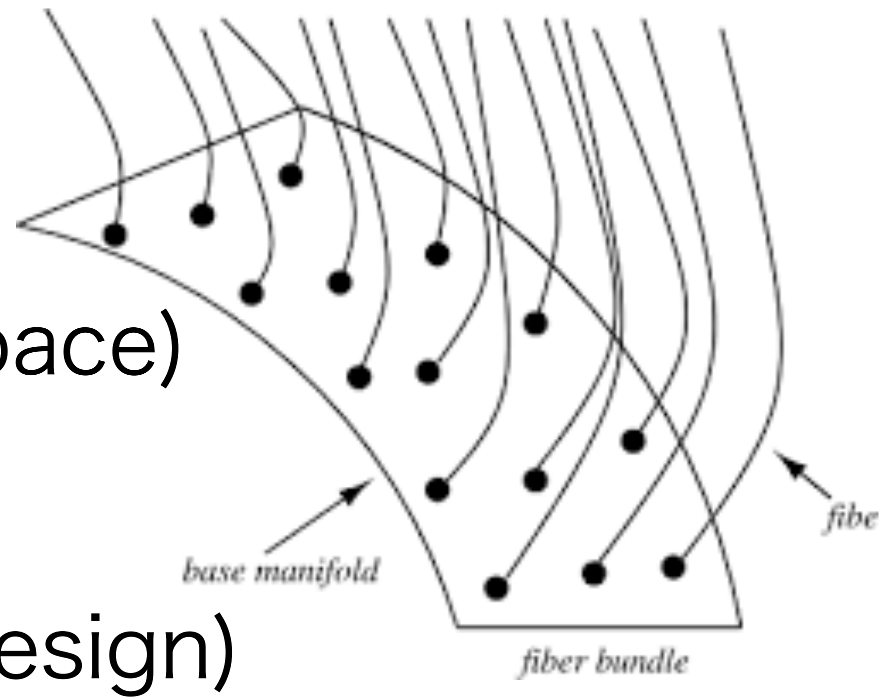
The direct product structure is realized only locally.
In general, it is "twisted" by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by E or $E \rightarrow X$

What is fiber bundle? Analogy for M1 students

X base space (space-time)
  = your head
F fiber (field)
  = your hair
E (= locally XxF) (total space)
  = your hair style
Connection
  = hair wax (local hair design)

Figure from Wolfram Math world



Classification of vector bundles

Let us consider the case F = some vector space.

Compare two vector bundles E_1 and E_2 .

It was proved that the homotopy theory can completely classify the vector bundles. But concrete computation is very difficult.

K-theory can classify the vector bundles when their rank is large enough, detecting some topological invariants to characterize the bundles with sophisticated computational techniques (more powerful than the standard (de Rham) cohomology theory with respect to characteristic classes).

What is K-theory?

- A mathematical theory which classifies the fiber (vector) bundles [or more general additive categories].
- One of generalized cohomology theories (stronger than ordinary cohomology) : without the dimension axiom:

$$H^{n>0}(\mathrm{point}) = \{0\}$$

- It is weaker than homotopy theory but easier to "compute".

$K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$
[ ] denotes the equivalence class (concrete definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \to E_2 \qquad\qquad D_{12}^\dagger : E_2 \to E_1$$

to represent the same element by $[E, D, \gamma]$
where

$$E = E_1 \oplus E_2, \;\; D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \;\; \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

* K^0 group describes classification of Dirac operator which anticommutes with chirality operator.

K-theory pushforward (Gysin map)

When you are interested in global structure only,
You can forget about details of the base manifold X by taking "one-point compactification" by the K-theory pushforward :

$$G : K^0(X) \to K^0(\text{point})$$

$$[E, D, \gamma] \to [H_E, D, \gamma]$$

The map just forgets all but the chiral symmetry.

H_E : The whole Hilbert space on which D acts.

Many information is lost but one (the Dirac operator index) remains.

Suspension isomorphism

“point” can be suspended to an interval:

There is an isomorphism between

$$K^0(\text{point}) \cong K^{-1}(I, \partial I)$$

$$[H_E, D, \gamma] \leftrightarrow [H_E \times I, D_t]$$

where “-1” denotes removal of the chirality operator.
Instead, the Dirac operator must become one-to-one (no zero mode) at the two endpoints : ∂I

Physical meaning of the isomorphism will be given soon later .

Bott periodicity theorem

Interestingly, we have another isomorphism
(Bott periodicity theorem) :

$$K^1(X,Y) \cong K^{-1}(X,Y)$$

"+1" adds a Clifford generator.

In the following, we simply denote it by K^1 .

In this talk, $K^1(I,\partial I)$ is the most important.

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Atiyah-Singer index

Index theorem

$$\underbrace{n_+ - n_-}_{\text{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\text{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[H_E, D, \gamma] \in K^0(\text{point})$

But we will show that it is isomorphic to

$$[H_E \times I, \gamma(D+m)] \in K^1(I, \partial I)$$

Eigenvalues of continuum massive Dirac operator

$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$ on Euclidean even-dimensional manifold.
Gauge group is U(1) or SU(N)

For $D_{\mathrm{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\mathrm{chirality}} m\phi.$

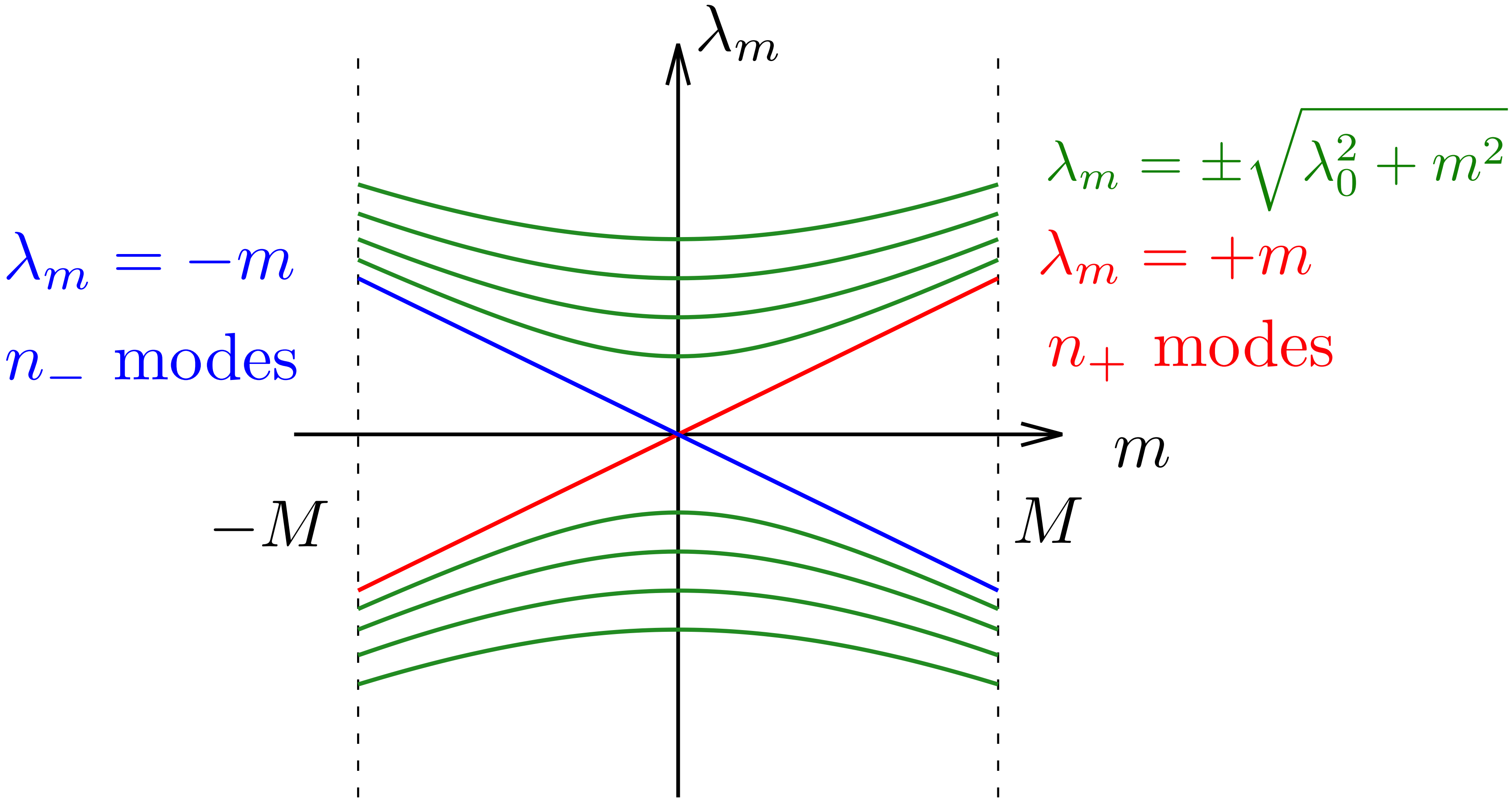
For $D_{\mathrm{cont.}}\phi \neq 0, \quad \{H(m), D_{\mathrm{cont.}}\} = 0.$

The eigenvalues are paired: $H(m)\phi_{\lambda_m} = \lambda_m\phi_{\lambda_m}$

$$H(m)D_{\mathrm{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\mathrm{cont.}}\phi_{\lambda_m}$$

As $H(m)^2 = -D_{\mathrm{cont.}}^2 + m^2$, we can write them $\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

Spectrum of $H(m) = \gamma_5(D_{\text{cont.}} + m)$



Spectral flow = Atiyah-Singer index = η invariant

n_+ = # of zero-crossing eigenvalues from - to +

n_- = # of zero-crossing eigenvalues from + to -

$$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$$

$n_+ - n_-$ =: spectral flow of $H(m)$ $\quad m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

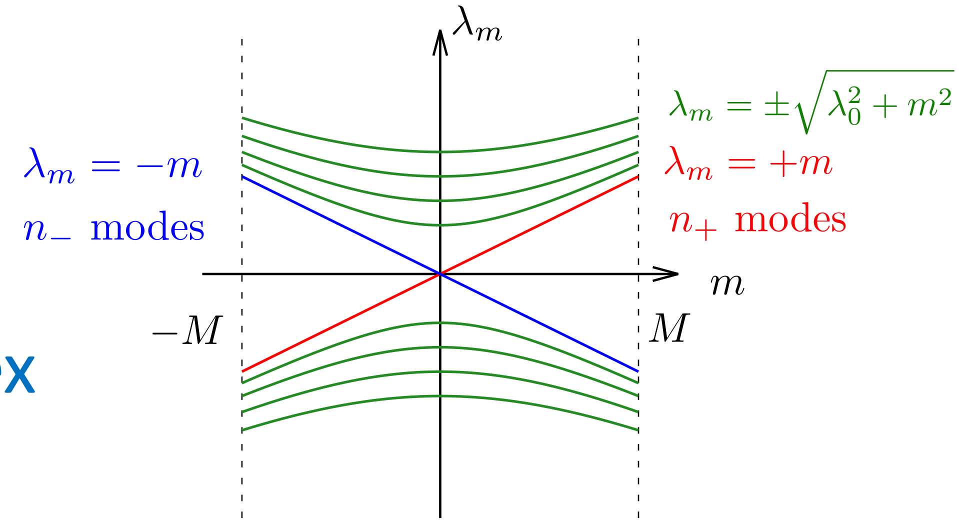
$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

Suspension isomorphism in K theory

Massless=
counting index
by points



Massive=
counting
index by lines

$$K^0(\text{point}) \cong K^{-1}(I, \partial I)$$

point — With chirality operator

line=interval — Without chirality operator

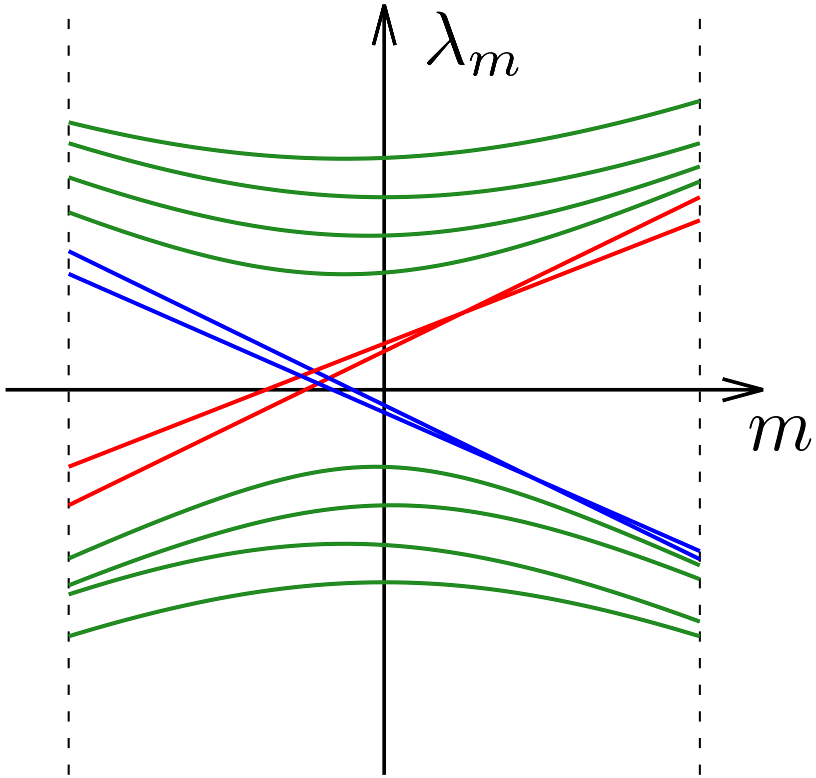
⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard definition:

Where is m=0?

What are zero modes?



Eta invariant:

If m= ± M points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

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Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus T^{2n}

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field A_i

D : Dirac operator on sections of E

$$D_{\mathrm{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality (Z_2 grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

Wilson Dirac operator on a lattice

We regularize T^{2n} is by a square lattice with lattice spacing a

(The fiber is still continuous.)

We denote the bundle by E^a and

link variables :

$$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right],$$

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f\nabla_i^b\right]$$

Wilson term

$$a\nabla_i^f\psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

Note: In our paper, we consider "generalized link variables" to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space I = range of mass [-M, M]

boundary ∂I = $\pm$M points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

D_m : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$[(\mathcal{H}, D_m)]$ having the same spectral flow.

Note: K^1 group does NOT require any chirality operator.

Definition of $K^1(I, \partial I)$ group

Group operation: $$[(\mathcal{H}^1, D^1_m)\} \pm \{(\mathcal{H}^2, D^2_m)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D^1_m & \\ & \pm D^2_m \end{pmatrix})]$$

Identity element: $$[(\mathcal{H}, D_m)]|_{\mathrm{Spec.flow}=0}$$

We compare $[(\mathcal{H}_{\mathrm{cont.}}, \gamma(D_{\mathrm{cont.}} + m))]$ and $[(\mathcal{H}_{\mathrm{lat.}}, \gamma(D_W + m))]$

taking their difference, and confirm if the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & f_a \\ f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

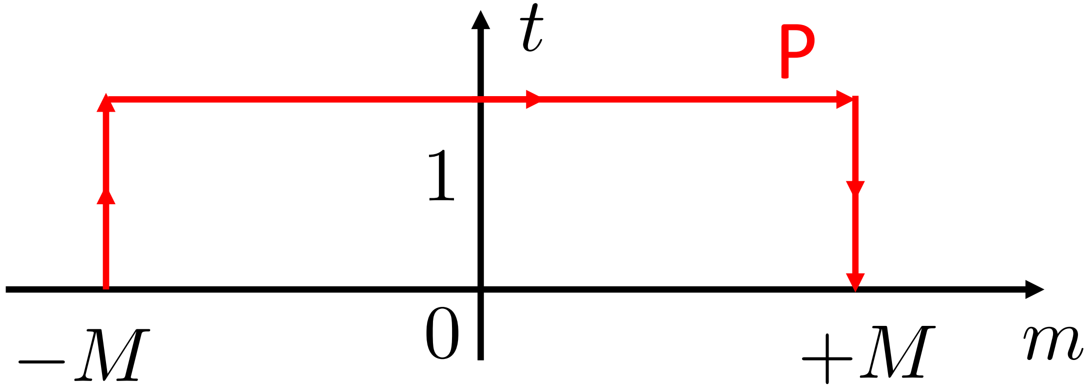
has Spectral flow =0 where f_a^* f_a are "mixing mass term" with some "nice" mathematical properties (see our paper for the details).

Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\rm cont.} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



Main theorem

There exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P
[which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D_{\text{cont.}} + m), \quad \gamma(D_W + m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D - M))^{\text{PV reg.}} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

Proof (by contradiction)

Assume $$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

Continuum limit

Multiplying $\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\text{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$$\hat{D}^2_\infty = D^2_{\text{cont.}} + m^2_\infty + t^2_\infty$$

requires

$$m_\infty = t_\infty = 0.$$

$u_\infty, \quad v_\infty$ are

L^2_1 weakly convergent

=

L^2 strongly convergent

(Rellich's theorem)

Contradiction with $m^2 + t^2 > 0$ along the path P.

Mathematical details

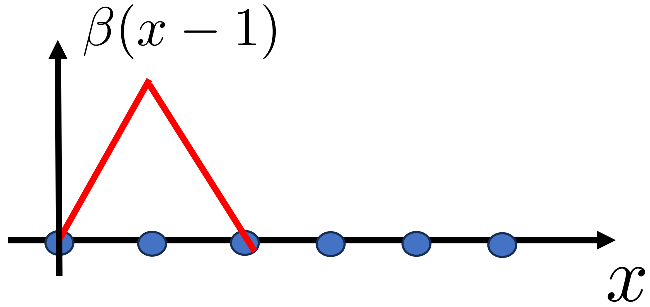
Because of time limitation, we may not be able to explain the followings.

- The map f_a, f_a^* between lattice and continuum Hilbert spaces
- Convergence of $f_a f_a^* \to 1, \quad f_a^* f_a \to 1.$
- Convergence of $f_a^* D_W f_a \to D_{\mathrm{cont.}}$
- Elliptic estimate for the Wilson Dirac operator
- Relich theorem

Please see our paper [S. Aoki, HF, M. Furuta, S. Matsuo, T. Onogi, S. Yamaguchi, **arXiv:2407.17708**]

f_a

$$f_a : H^{\text{lat.}} \to H^{\text{cont.}}$$



From finite-dimensional vector bundle on a discrete lattice we need to make infinite-dimensional vector bundle on continuous x :

$$f_a \phi^{\text{lat.}}(x) = \sum_{l \in C_x} \beta(x-l) P(x-l) \phi^{\text{lat.}}(l)$$

C_x : a hyper cube containing x . l : lattice sites

$$P(x-l) = P \exp\left[i \int_l^x dx'^i A_i(x') \right]$$ Wilson line.

$\beta(x-l)$: linear partition of unity s.t.

$$\beta(0) = 1, \beta(\pm a e_\mu) = 0, \quad \sum_{l \in C_x} \beta_l(x) = 1.$$

$$f_a^*$$

$$f_a^* : H^{\mathrm{cont.}} \to H^{\mathrm{lat.}}$$

Is defined by

$$f_a^* \phi^{\mathrm{cont.}}(l) = \int_{y \in C_l} dy \beta(l-y) P(l-y) \phi^{\mathrm{cont.}}(y)$$

Note) $f_a^* f_a$ is not the identity but smeared to nearest-neighbor sites. (The gauge invariance is maintained by the Wilson lines.)

Continuum limit of f_a^* f_a

1. For arbitrary $\phi^{\mathrm{lat.}}$

   $\lim_{a\to 0} f_a\phi^{\mathrm{lat.}}$ weakly converges to a $\phi_0^{\mathrm{cont.}} \in L_1^2$

   where L_1^2 is the square-integrable subspace of $H^{\mathrm{cont.}}$

   to the first derivatives.

2. $\lim_{a\to 0} f_a\gamma(D_W + m)\phi^{\mathrm{lat.}}$ weakly converges to

   $\gamma(D+m)\phi_0^{\mathrm{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a\phi^{\mathrm{lat.}} - \phi^{\mathrm{lat.}}||_{L^2}^2 < ca^2||\phi^{\mathrm{lat.}}||_{L_1^2}^2$

4. For any $\phi^{\mathrm{cont.}} \in L_1^2$, $\lim_{a\to 0} f_a f_a^*\phi^{\mathrm{cont.}}$

converges to $\phi_0^{\mathrm{cont.}} \in L_1^2$ and $\lim_{a\to 0} f_a f_a^*\phi_0^{\mathrm{cont.}} = \phi_0^{\mathrm{cont.}}$

Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,
a constant c exists such that

$$||D_i\phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large、D is also large.
This property is nontrivial on a lattice.

$$||\nabla_i^f\phi||^2 \leq c(||\phi||^2 + ||D_W\phi||^2)$$

Doubler modes have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important, too!

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Counting lines (massive, K^1) is easier than counting points (massless, K^0).

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The proof is given by lattice-continuum combined Dirac operator, which is gapped.

6. Comparison with the overlap Dirac index

7. Summary and discussion

Wilson Dirac operator is equally good as D_{ov} to describe the index.

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

By $K^1(I, \partial I)$ for sufficiently small lattice spacings

Suspension isomorphism

Wilson Dirac operator is equally good as D_{ov} to describe the index.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

By $K^1(I, \partial I)$ for sufficiently small lattice spacings

Suspension isomorphism

Or even better?

Application to the manifolds with boundaries

Periodic b.c.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

Dirichlet b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D_{DW}^{\mathrm{cont.}})) = \mathrm{Ind}_{\mathrm{APS}} D^{\mathrm{cont.}}$$

[perturbative equality F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019].

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

But the overlap Dirac is missing because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I, \partial I) \Rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D - M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we will be able to show

$$KO^1(I, \partial I) \Rightarrow -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_W - M}{D_W + M}\right)\right] = -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_{\mathrm{cont.}} - M}{D_{\mathrm{cont.}} + M}\right)\right]$$

$$= \mathrm{Ind}_{\mathrm{mod-two}} D_{\mathrm{cont.}}$$

But there is no overlap Dirac counterpart.

Contents

✓ 1. Introduction

We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)

is great but equivalent to the eta invariant of the massive Wilson Dirac op.

✓ 3. K-theory

classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

✓ 4. Massless Dirac (K^0 group) vs. massive Dirac (K^1 group) in continuum

Counting lines (massive, K^1) is easier than counting points (massless, K^0).

✓ 5. Main theorem on a lattice

The proof is given by lattice-continuum combined Dirac operator, which is gapped.

✓ 6. Comparison with the overlap Dirac index

we expect wider applications (to APS boundary and real case) than the overlap index.

7. Summary and discussion

Summary

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…]

Summary

$$K^1(I, \partial I)$$

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

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Backup slides

What are the weak convergence and strong convergence?

The sequence v_j weakly converges to v_∞

when for arbitrary w

$$\lim_{j\to\infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j\to\infty} (v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich's theorem:

L^2_1 weak convergence = L^2 convergence