# The index of lattice Dirac operators and K-theory

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Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), "The index of lattice Dirac operators and K-theory," arXiv:2407.17708, 2501.02873

# Let me introduce myself first.

My major is particle physics, in particular, numerical simulation of lattice QCD(quantum chromo-dynamics).

During 2009-2010 I was an assistant professor here in Nagoya U. specially appointed for the GCOE program "Quest for Fundamental Principles in the Universe".

At the interview, Prof. Naoshi Sugiyama asked me that

"In this GCOE program, please study something interdisciplinary among different fields."

I tried but I could not write any papers, which was my regret.

But after >10 years, I finally found something interdisciplinary with mathematicians and wrote some papers. This talk on a work using K-theory is one of them.

# What is the index of Dirac operators ?

$$D\psi = 0 \quad D := \gamma^\mu(\partial_\mu + iA_\mu)$$

we consider
U(1) or SU(N) group

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) = \boldsymbol{E}\cdot\boldsymbol{B}$$

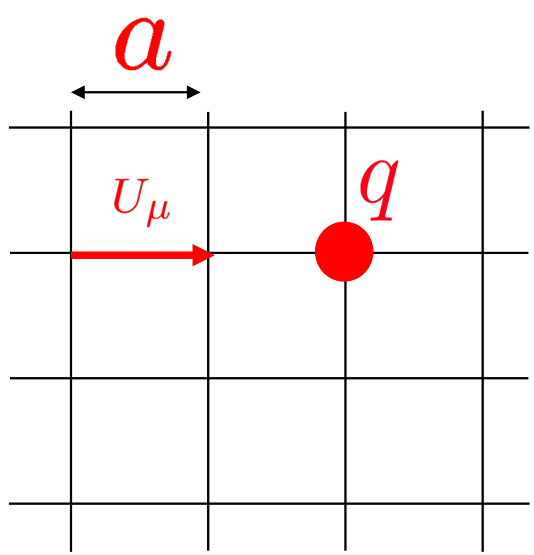
Topological charge

#sol with + chirality

#sol with - chirality

Important both in physics and mathematics to understand gauge field topology, which is non-perturbative.

# What is lattice gauge theory?




It is a (non-perturbative) regularization of quantum field theory with lattice spacing $a$

Gauge fields(gluons) live on links

$$U_{n,\mu} = \exp(igaA_\mu(n+\hat{\mu}/2))$$

Fermions (quarks) live on sites $q_n = q(n)$

The Lagrangian is given by for example,

$$L = \beta \sum_{\mu,\nu=1}^{4} \mathrm{Tr}[U_{n,\mu}U_{n+\mu,\nu}U^\dagger_{n+\nu,\mu}U^\dagger_{n,\nu}] + \bar{q}_n \left[ \sum_\mu \gamma_\mu \frac{U_{n,\mu}q_{n+\hat{mu}} - U^\dagger_{n-\hat{mu},\mu}q_{n-\hat{\mu}}}{2a} + m \right] q_n$$

which converges to QCD Lagrangian in the $a \to 0$ limit.

# Our goal

= A mathematical formulation of the index (theorem) on a lattice.

In continuum, Dirac operator is a differential operator.

$$D\psi = \gamma^\mu(\partial_\mu + iA_\mu)\psi.$$

On lattice, Dirac operator is a difference operator.

$$D\psi = \gamma^\mu[U_\mu(x)\psi(x+\mu a) - \psi(x)]/a.$$

Mathematically nontrivial.

[Related works by mathematicians: Kubota 2020, Yamashita 2021]

# Difficulty in lattice gauge theory

Both of Dirac index and topology are difficult on the lattice:

- It is difficult to define the chiral zero modes, since the standard lattice Dirac operators break the chiral symmetry.

- Lattice discretization of space time makes the topology not well-defined.

# A traditional solution = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] satisfying the Gingparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$$

a modified chiral symmetry is exact [Luescher 1998],

and the index is well-defined: $\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$

[Hasenfratz et al. 1998]

but this definition is so far limited to the even-dimensional flat periodic lattice.

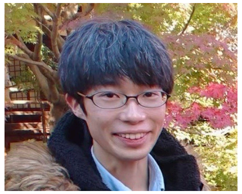
## This work = an alternative mathematical formulation of the lattice Dirac index.

In our formulation,

- No exact chiral symmetry is needed : the standard Wilson Dirac operator is good enough.
- K theory is used to show equality to the continuum Dirac index.
- Wider application than the overlap Dirac operator to the systems with nontrivial boundaries in any dimensions.

# Phys-Math collaborators

Physicists



Shoto Aoki

Tetsuya Onogi

Satoshi Yamaguchi

Mathematicians

Mikio Furuta

Shinichiroh Matsuo

# Physicist-friendly Dirac index project

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two index version [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]
- Lattice version (without boundary) [Aoki, F, Furuta, Matsuo, Onogi, Yamaguchi 2024] = this work.

Q. How physicist-friendly?
A. We do not need to take care of chiral symmetry and unphysical boundary conditions in our formulation.

# Contents

# Continuum-> Lattice : derivative -> difference

Continuum Dirac operator

$$D\psi(x) = \gamma^\mu(\partial_\mu)\psi(x) = \int dp\gamma^\mu(ip_\mu)\tilde{\psi}(p)e^{ipx}$$

(A naïve) lattice Dirac operator

$$D\psi(x) = \gamma^\mu \frac{\psi(x+\hat{\mu}a) - \psi(x-\hat{\mu}a)}{2a} = \int dp\gamma^\mu \frac{e^{ip(x+\hat{\mu}a)} - e^{ip(x-\hat{\mu}a)}}{2a}\tilde{\psi}(p)$$

$$= \int dp\gamma^\mu i\frac{\sin p_\mu a}{a}\tilde{\psi}(p)e^{ipx}.$$

$a$ :lattice spacing

$\hat{\mu}$ : unit vector in μ direction.

which has zero points at $p_\mu = 0, \quad \frac{\pi}{a}$

(phys) Doublers appear!

(math) Ellipticity is lost!

# Wilson Dirac operator

$a$ :lattice spacing

$\hat{\mu}$ : unit vector in μ direction.

The Wilson Dirac operator is commonly used in lattice gauge theory.

$$D_W = \sum_\mu \left[ \gamma^\mu \frac{\nabla^f_\mu + \nabla^b_\mu}{2} - \frac{a}{2} \nabla^f_\mu \nabla^b_\mu \right]$$

$$\nabla^f \psi(x) = \frac{\psi(x+\hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^b \psi(x) = \frac{\psi(x) - \psi(x-\hat{\mu}a)}{a}$$

The additional term corresponds the Laplacian and the Fourier transformation

$$\sum_\mu \gamma^\mu i \frac{\sin p_\mu a}{a} + \sum_\mu \frac{(1-\cos p_\mu a)}{a}$$

= Large mass term

Except for $p_\mu = 0$

indicates that the doublers cannot excite (recovering ellipticity)due to heavy mass but chiral symmetry ($Z_2$ grading) is lost:

$$\gamma_5 D_W + D_W \gamma_5 \neq 0.$$

# Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:
If $\gamma_5 D + D\gamma_5 = 0,$ we cannot avoid fermion doubling.

(we have to give up $Z_2$ grading to recover ellipticity)

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

can avoid NN theorem.
But no concrete form was found in ~20 years.

# Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1+\gamma_5 \mathrm{sgn}(H_W)\right) \quad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$

$$\gamma_5(1-aD_{ov}/2)\gamma_5 D_{ov} + \gamma_5 D_{ov}\gamma_5(1-aD_{ov}/2) = 0.$$

$\Rightarrow$ $\Gamma_5 H + H\Gamma_5 = 0.$

$$H = \gamma_5 D_{ov}, \quad \Gamma_5 = \gamma_5\left(1-\frac{aD_{ov}}{2}\right)$$

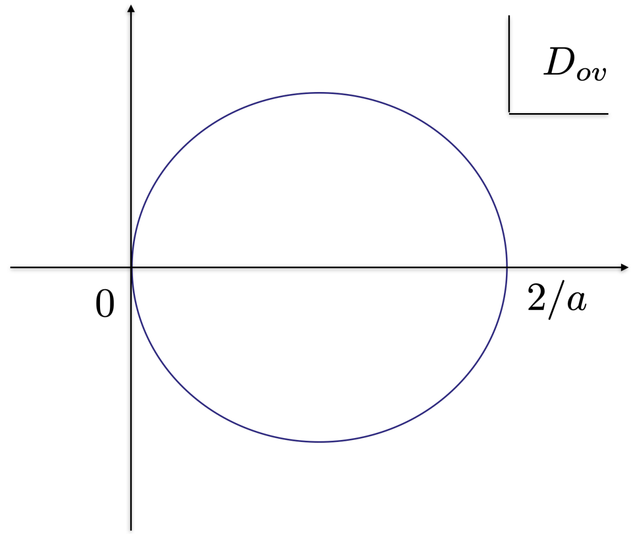
= a modified exact chiral symmetry but $\Gamma_5^2 \neq 1.$

[Luescher 1998]

# We can define the index !

[Hasenfratz et al. 1998]

$a$ :lattice spacing




Overlap Dirac spectrum lies on a circle with radius 1/a

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)\psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real 2/a (doubler poles) do not contribute.

$$\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\text{zero-modes}}\gamma_5 = n_+ - n_-$$

But $D_{ov}$ is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\begin{aligned}\mathrm{Ind} D_{ov} &= \mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) \\ &= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)\end{aligned}$$

But $D_{ov}$ is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

What is this ???

# η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}}\mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

# The Wilson Dirac operator and K-theory

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

In this talk, we try to show a deeper mathematical meaning of the right-hand side of the equality,

and try to convince you that the massive Wilson Dirac operator is an equally good or even better object than $D_{ov}$ to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…]

# Contents

✓ 1. Introduction

We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)

is great but equivalent to the eta invariant of the massive Wilson Dirac op.

3. K-theory

4. Massless Dirac ($K^0$ group) vs. massive Dirac ($K^1$ group) in continuum

5. Main theorem on a lattice

6. Comparison with the overlap Dirac index

7. Summary and discussion

# What is fiber bundle (for physicists)?

A united manifold of spacetime (= base manifold) and field (fiber)

$$\phi(x) \ \rightarrow (x, \phi) \in X \times F$$

Spacetime
= base space

Field space
= fiber space

The direct product structure is realized only locally.
In general, it is "twisted" by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by $E$ or $E \rightarrow X$

# What is fiber bundle? Analogy for (phys) students

X base space (space-time)

= your head

F fiber (field)

= your hair

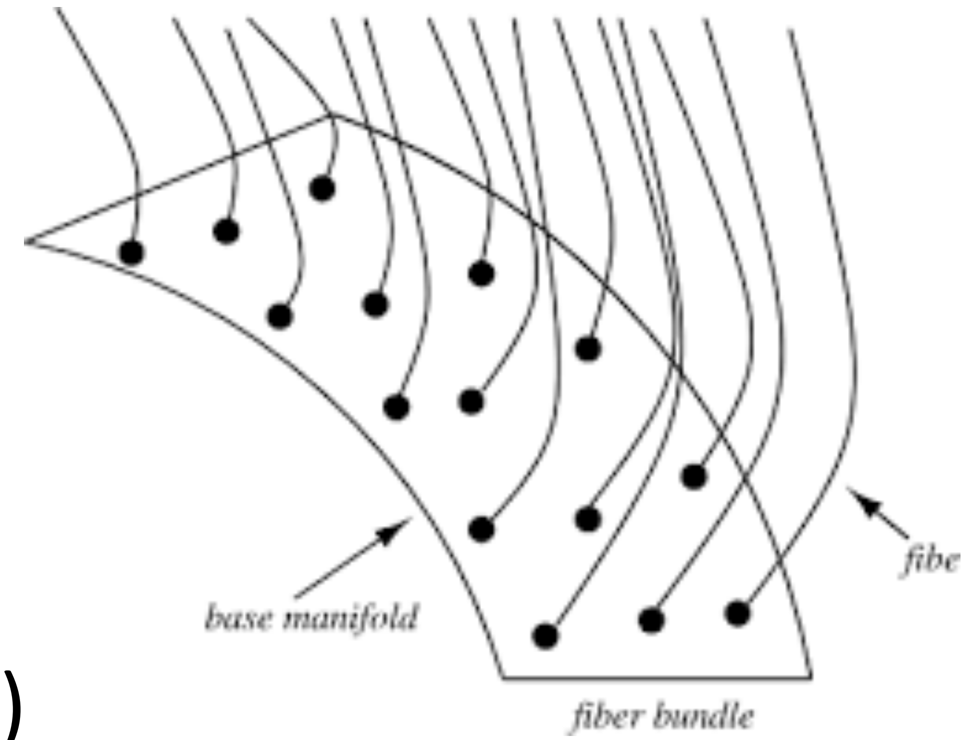
E (= locally XxF) (total space)

= your hair style

Connection

= hair wax (local hair design)

Figure from Wolfram Math world



# Classification of vector bundles

Let us consider the case F = some vector space.

Compare two vector bundles $E_1$ and $E_2$.

It was proved that the homotopy theory can completely classify the vector bundles. But concrete computation is very difficult.

K-theory can classify the vector bundles when their rank is large enough, detecting some topological invariants.
(more powerful than the standard (de Rham) cohomology theory).

# $K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$

[ ] denotes the equivalence class (concrete definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \to E_2 \qquad D_{12}^\dagger : E_2 \to E_1$$

* To be precise, D acts on the sections of E.

to represent the same element by $[E, D, \gamma]$

where

$$E = E_1 \oplus E_2, \quad D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

* $K^0$ group describes classification of Dirac operator which anticommutes with chirality operator.

# K-theory pushforward

When we are interested in global structure only,
We can forget about details of the base manifold X by taking
"one-point compactification" or the K-theory pushforward :

$$G : K^0(X) \to K^0(\text{point})$$

$$[E, D, \gamma] \to [H_E, D, \gamma]$$

The map just forgets all but the chiral symmetry.

$H_E$ : The whole Hilbert space on which $D$ acts.

Many information is lost but one (the Dirac operator index) remains.

# Suspension isomorphism

The "point" can be suspended to an interval:

There is an isomorphism between

$$K^0(\text{point}) \cong K^{-1}(I, \partial I)$$

$$[H_E, D, \gamma] \leftrightarrow [p^* H_E, D_t]$$

$p^*$ : pull-back of $p : I \to \text{point}.$
we omit in the following.

where the superscript "-1" reflects removal of the chirality operator. Instead, the Dirac operator must become one-to-one (no zero mode) at the two endpoints : $\partial I$

Physical meaning of the isomorphism will be given soon later .

# Bott periodicity theorem

We have another isomorphism
(Bott periodicity theorem) :

$$K^1(X,Y) \cong K^{-1}(X,Y)$$

"+1" adds a Clifford generator.

In the following, we simply denote it by $K^1$.

In this talk, $K^1(I,\partial I)$ is the most important.

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# Atiyah-Singer index

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[H_E, D, \gamma] \in K^0(\mathrm{point})$

But we will show that it is isomorphic to

$$[H_E, \gamma(D+m)] \in K^1(I, \partial I)$$

# Eigenvalues of continuum massive Dirac operator

$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$

on Euclidean even-dimensional manifold.
Gauge group is U(1) or SU(N)

For $D_{\mathrm{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\mathrm{chirality}} m\phi.$

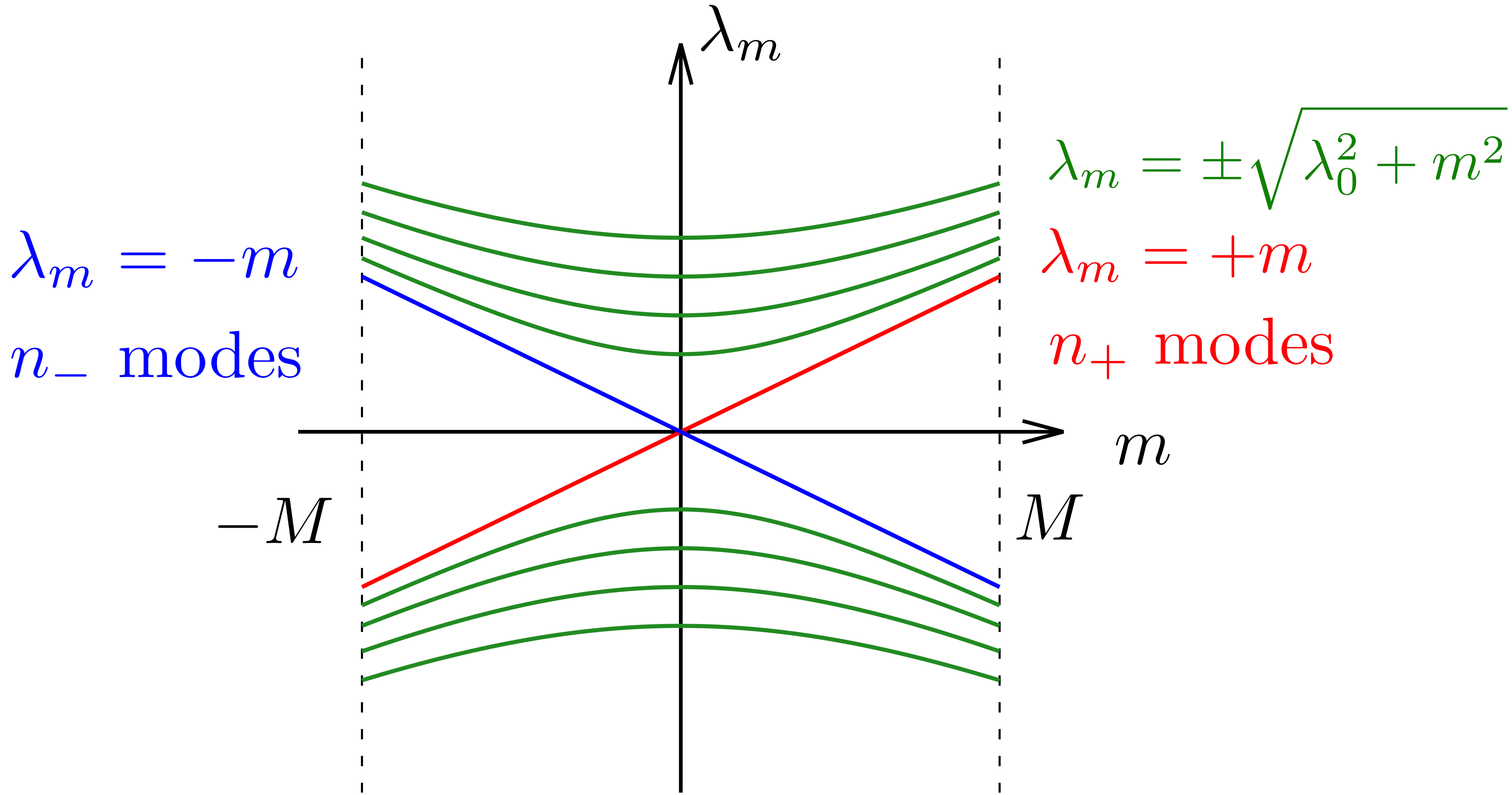
For $D_{\mathrm{cont.}}\phi \neq 0, \quad \{H(m), D_{\mathrm{cont.}}\} = 0.$

The eigenvalues are paired: $H(m)\phi_{\lambda_m} = \lambda_m \phi_{\lambda_m}$

$$H(m)D_{\mathrm{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\mathrm{cont.}}\phi_{\lambda_m}$$

As $H(m)^2 = -D_{\mathrm{cont.}}^2 + m^2$ , we can write them $\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

# Spectrum of $H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$



# Spectral flow = Atiyah-Singer index = η invariant

$n_+$ = # of zero-crossing eigenvalues from - to +

$n_-$ = # of zero-crossing eigenvalues from + to -

$$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$$

$n_+ - n_-$ =: spectral flow of $H(m)$ $\quad m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

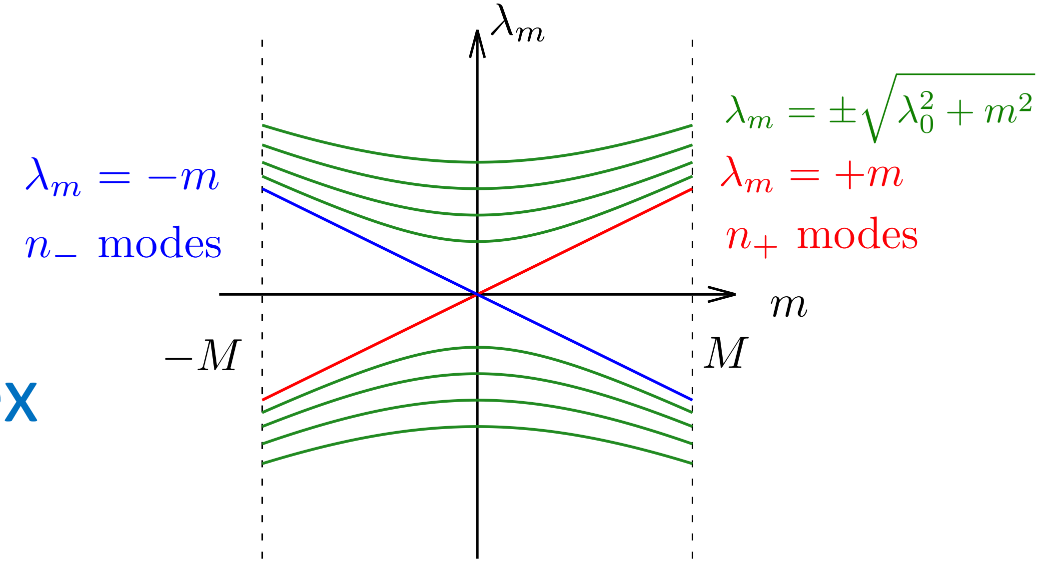
$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

# Suspension isomorphism in K theory

Massless= counting index by points




Massive= counting index by lines

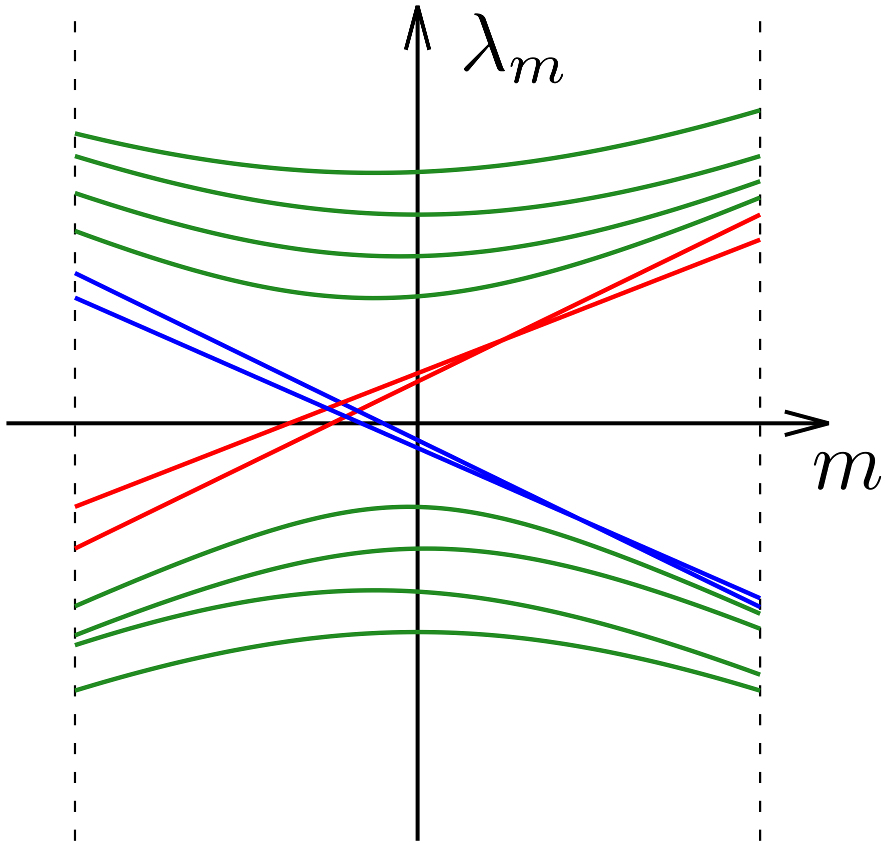
$$K^0(\text{point}) \cong K^1(I, \partial I)$$

| | |
|---|---|
| point | line=interval |
| with chirality operator | without chirality operator |

⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard definition:
Where is m=0?
What are zero modes?




Eta invariant:
If m= $\pm$ M points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

# Contents

# Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus $T^{2n}$

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field $A_i$

D : Dirac operator on sections of E

$$D_{\mathrm{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality ($Z_2$ grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

# Wilson Dirac operator on a lattice

We regularize $T^{2n}$ is by a square lattice with lattice spacing $a$

(The fiber is still continuous.)

We denote the bundle by $E^a$ and

link variables :

$$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right],$$

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f \nabla_i^b\right]$$

Wilson term

$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

Note: In our paper, we consider ”generalized link variables” to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

# Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space $I$ = range of mass [-M, M]

boundary $\partial I$ = $\pm$M points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

$D_m$ : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$$[(\mathcal{H}, D_m)]$$ having the same spectral flow.

Note: $K^1$ group does NOT require any chirality operator.

# Definition of $K^1(I, \partial I)$ group

Group operation: $[(\mathcal{H}^1, D^1_m)\} \pm \{(\mathcal{H}^2, D^2_m)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D^1_m & \\ & \pm D^2_m \end{pmatrix})]$

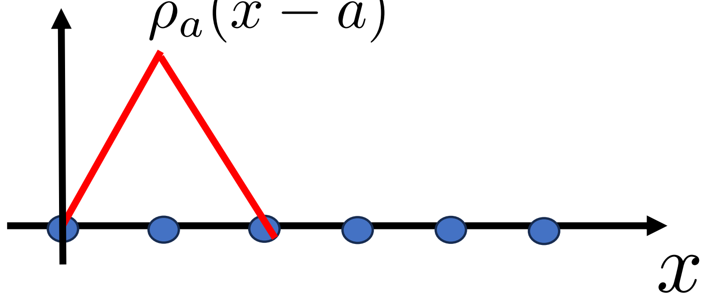
Identity element: $[(\mathcal{H}, D_m)]|_{\mathrm{Spec.flow}=0}$

We compare $[(\mathcal{H}_{\mathrm{cont.}}, \gamma(D_{\mathrm{cont.}} + m))]$ and $[(\mathcal{H}_{\mathrm{lat.}}, \gamma(D_W + m))]$ taking their difference, and confirm if the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & f_a \\ f^*_a & -\gamma(D_W + m) \end{pmatrix}$$

has Spectral flow =0 where $f^*_a$ $f_a$ are "mixing mass term" with some "nice" mathematical properties.

$$f_a : H^{\text{lat.}} \to H^{\text{cont.}}$$




maps from finite-dimensional Hilbert space on a discrete lattice to infinite-dimensional continuum one :

$$f_a\phi(x) := a^n \sum_{z\in\text{lattice sites}} \rho_a(x-z)U(x,z)\phi(z).$$

$U(x,z)$ : parallel transport (or Wilson line) to ensure the gauge invariance.

$\rho_a(x-z)$: weight function (multi-) linearly interpolating the nearest-neighbors.

To control the norm before/after the map, it satisfies

$$\int_{x\in T^n} \rho_a(x-z)d^n x = 1 \qquad a^n \sum_{z\in\text{lattice sites}} \rho_a(x-z) = 1.$$

$$f_a^* : H^{\mathrm{cont.}} \to H^{\mathrm{lat.}}$$

Is defined by

$$f_a^* \psi_1(z) := \int_{x \in T^n} \rho_a(z-x) U(x,z)^{-1} \psi_1(x) d^n x.$$

Note) $f_a^* f_a$ is not the identity but smeared around nearest-neighbor sites.
(The gauge invariance is maintained by the Wilson lines.)

# Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,

a constant c exists such that

$$||D_i\phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large, D is also large.

This property is nontrivial on a lattice.

$$||\nabla_i^f \phi||^2 \leq c(||\phi||^2 + ||D_W\phi||^2)$$

Without Wilson term, doubler modes would have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important to make the Dirac operator elliptic.

# Continuum limit of $f_a^* \quad f_a$

1. For arbitrary $\phi^{\mathrm{lat.}}$

   $\lim_{a\to 0} f_a \phi^{\mathrm{lat.}}$ weakly converges to a $^{\exists}\phi_0^{\mathrm{cont.}} \in L_1^2$

   where $L_1^2$ is a subspace of $H^{\mathrm{cont.}}$ where the elements and their first derivatives are square integrable.

2. $\lim_{a\to 0} f_a \gamma(D_W + m)\phi^{\mathrm{lat.}}$ weakly converges to $\gamma(D+m)\phi_0^{\mathrm{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a \phi^{\mathrm{lat.}} - \phi^{\mathrm{lat.}}||_{L^2}^2 < ca^2 ||\phi^{\mathrm{lat.}}||_{L_1^2}^2$

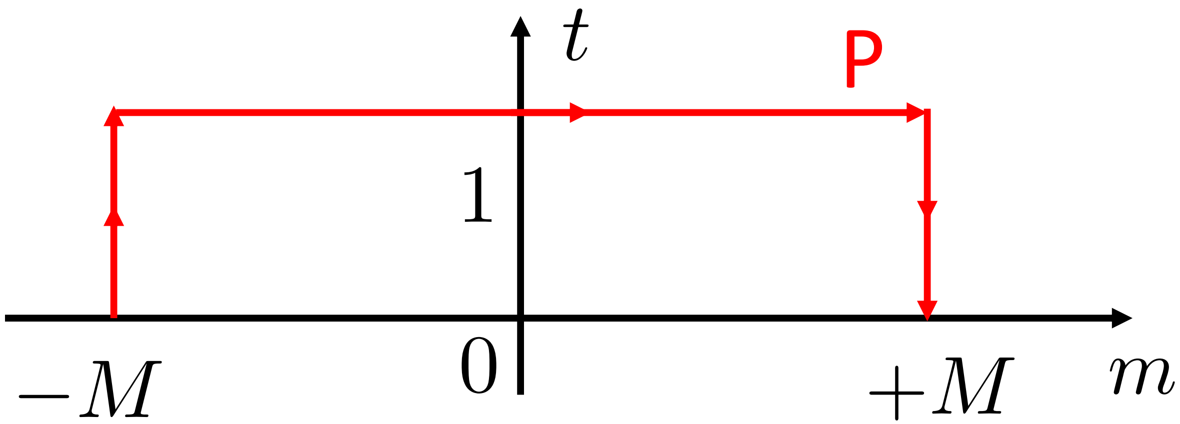
4. For any $\phi^{\mathrm{cont.}} \in L_1^2$, $\lim_{a\to 0} f_a f_a^* \phi_0^{\mathrm{cont.}} = \phi_0^{\mathrm{cont.}}$

# Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



# Main theorem

There exists a finite lattice spacing $a_0$ such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P

[which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D_{\text{cont.}} + m), \ \ \gamma(D_W + m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D - M))^{\text{PV reg.}} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

# Proof (by contradiction)

Assume
$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

# Continuum limit

Multiplying $\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\mathrm{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$$\hat{D}_\infty^2 = D_{\mathrm{cont.}}^2 + m_\infty^2 + t_\infty^2$$

requires

$$m_\infty = t_\infty = 0.$$

$u_\infty, \quad v_\infty$ are

$L_1^2$ weakly convergent

=

$L^2$ strongly convergent

(Rellich's theorem)

Contradiction with $m^2 + t^2 > 0$ along the path P.

# Numerical test

We consider a two-dimensional square lattice (or torus)
We set link variables as

$$U_y(x, y) = \exp\left[i\frac{2\pi Q(x - x_0)a}{L_1^2}\right]$$

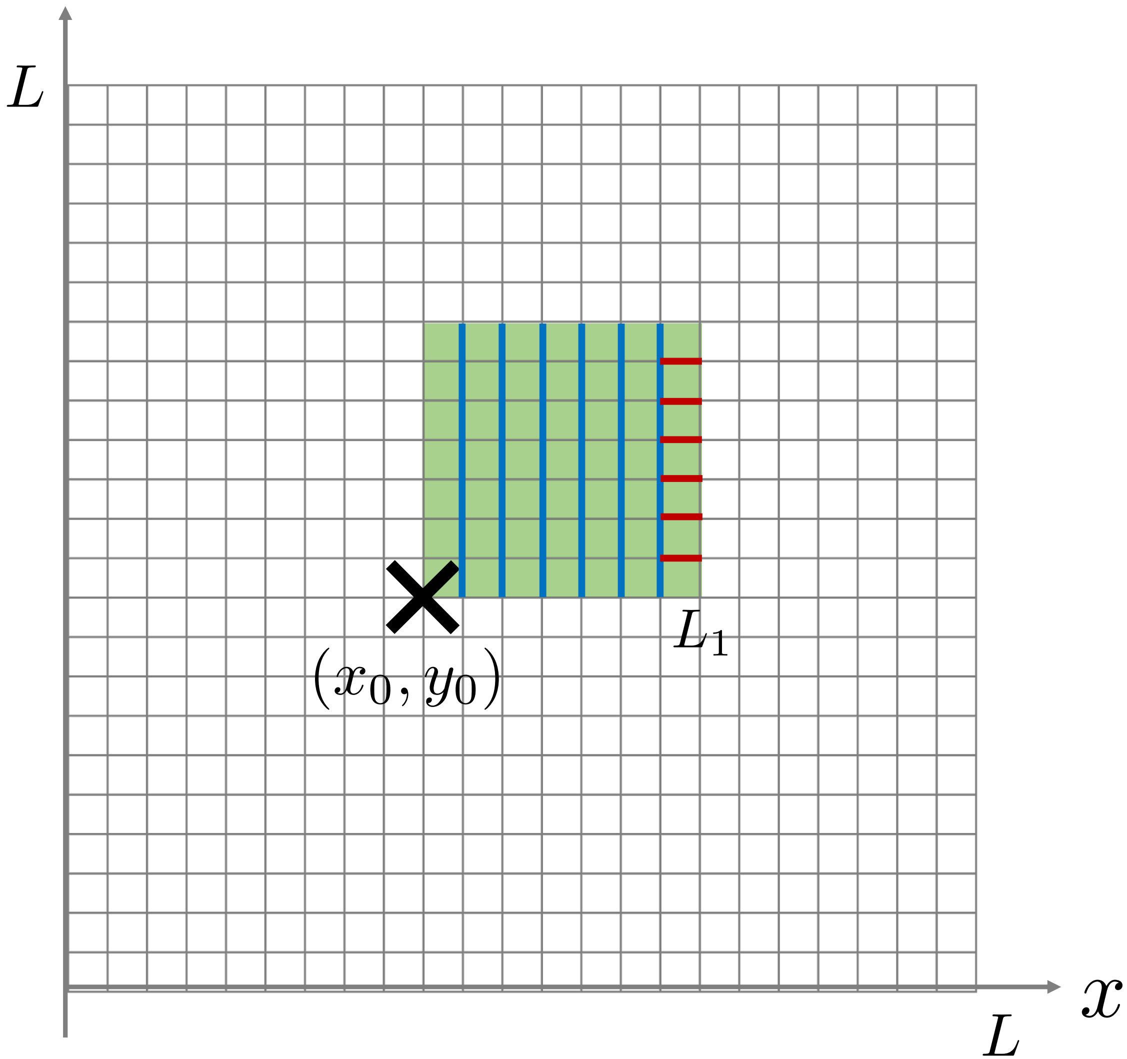
$$U_x(x, y) = \exp\left[-i\frac{2\pi Q(y - y_0)}{L_1}\right]$$

$$\text{others} = 1.$$

Then every green plaquette has a constant curvature

$$U_P(x, y) = \exp\left[i\frac{2\pi Q a^2}{L_1^2}\right]$$

so that geometrical index will be Q.




This constant curvature background can be extended to any even dimensions with SU(N) gauge connections [Cf. Hamanaka-Kajiura 2002].

# Massive Wilson Dirac

$$\gamma D_W(m) = \gamma \left[ \sum_i \left[ \gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right] + m \right]$$

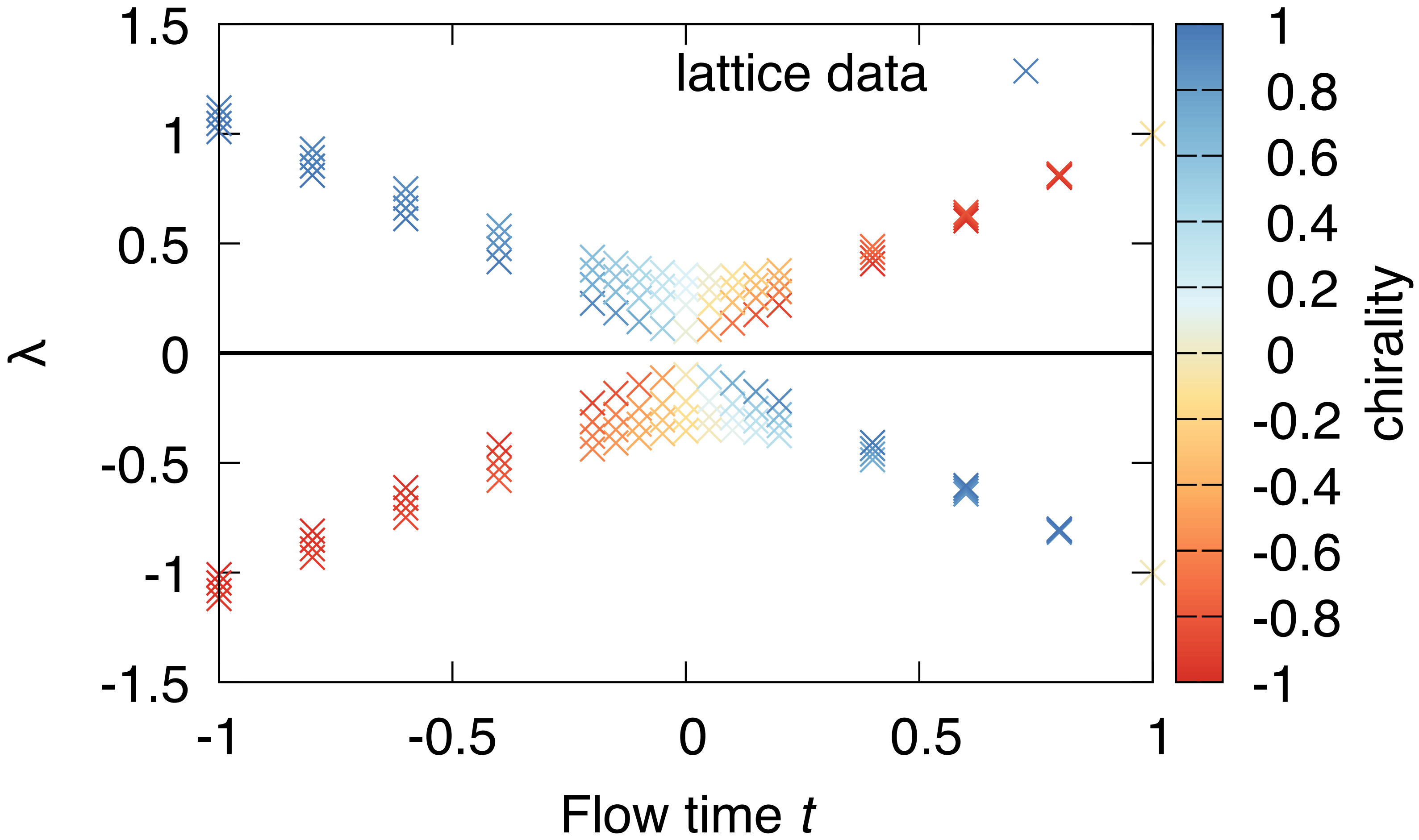
$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x}) \qquad a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i$$

with periodic b.c. in x-direction and anti-periodic b.c. in y direction. We set L=32 and L1=10.

We compute near-zero eigen-spectrum

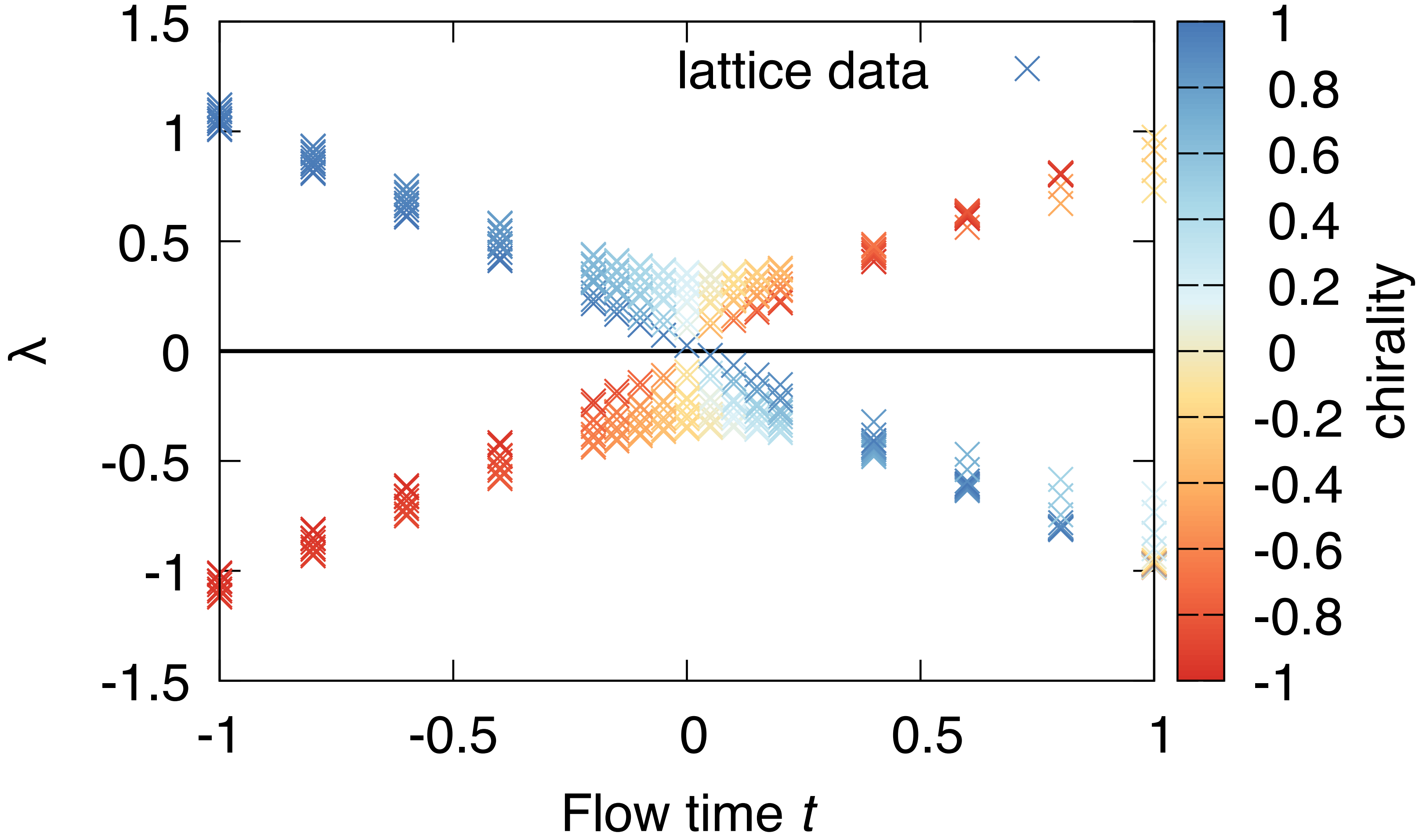
in the range $-1 \leq m \leq +1$

# Numerical result at Q=0



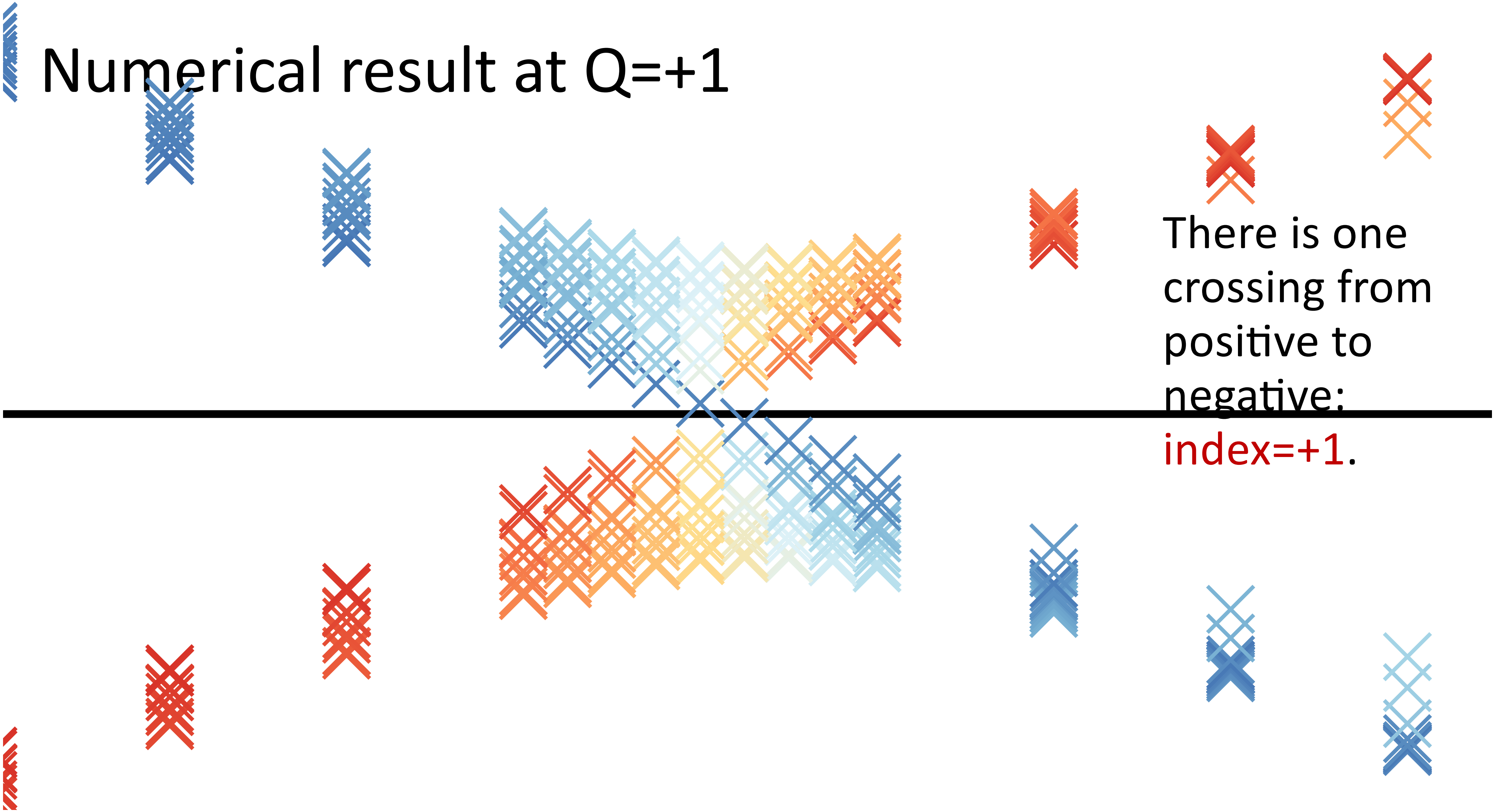

There is no zero crossing : index=0.

# Numerical result at Q=+1



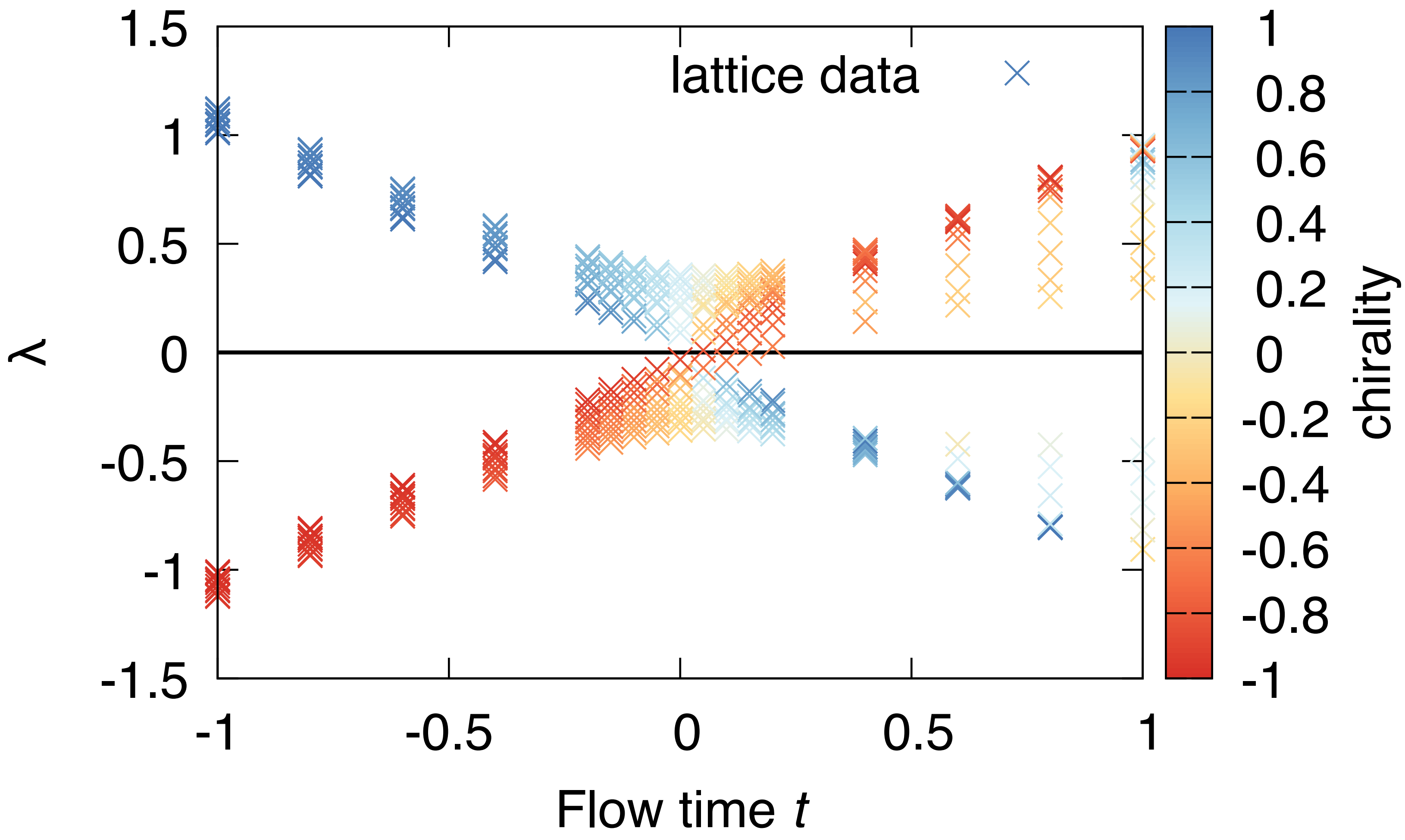

There is one crossing from positive to negative: index=+1.

# Numerical result at Q=+1



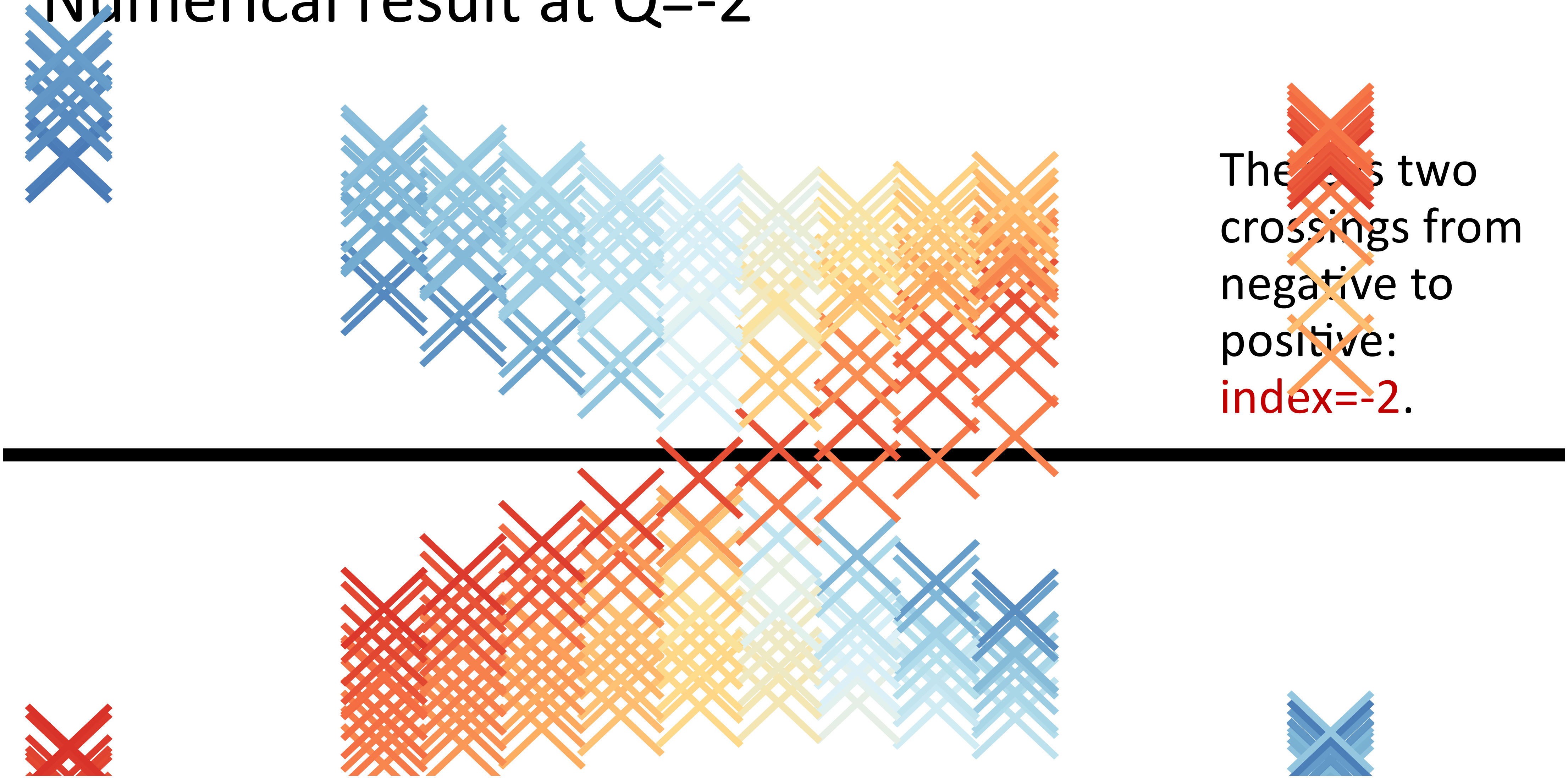
There is one crossing from positive to negative: index=+1.

# Numerical result at Q=-2



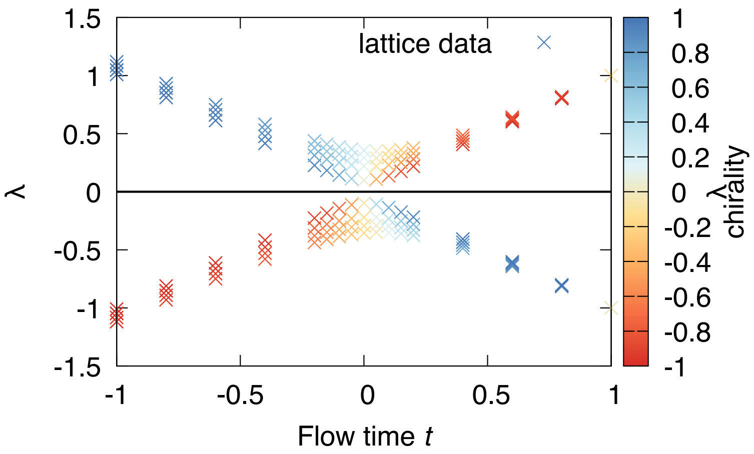

There is two crossings from negative to positive: index=-2.

# Numerical result at Q=-2

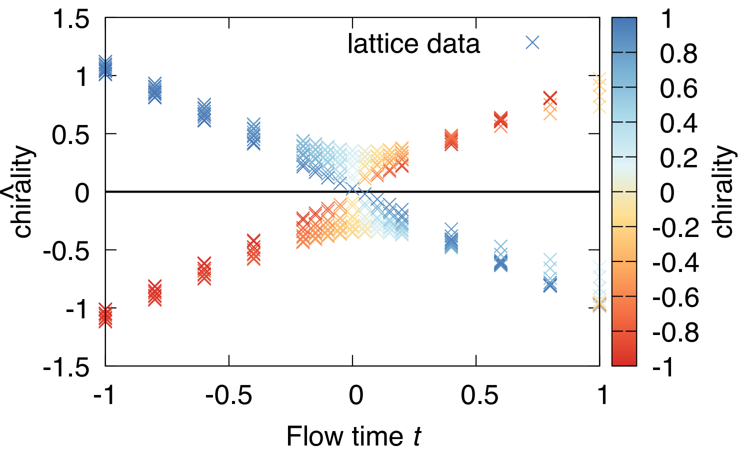


There is two crossings from negative to positive: index=-2.

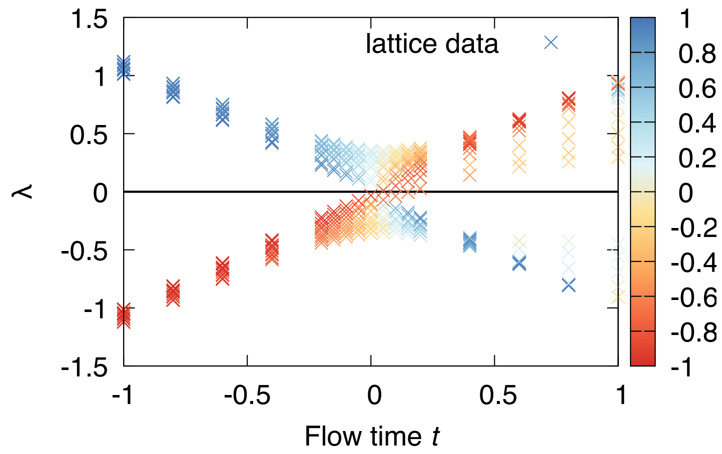
# Our 32x32 lattice reproduces the Atiyah-Singer index theorem on a torus.




Index=Q=0




Index=Q=+1




Index=Q=-2

# Contents

Wilson Dirac operator is equally good as $D_{ov}$ to describe the index.

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

By $K^1(I, \partial I)$
for sufficiently small lattice spacings

Suspension isomorphism

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By $K^1(I, \partial I)$ for sufficiently small lattice spacings

Suspension isomorphism

Or even better?

# Application to the manifolds with boundaries

Periodic b.c.

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

Dirichlet b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D_{DW}^{\mathrm{cont.}})) = \mathrm{Ind}_{\mathrm{APS}}D^{\mathrm{cont.}}$$

[perturbative equality F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019].

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

But the overlap Dirac is missing because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

# Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I, \partial I) \Rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D - M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we will be able to show

$$KO^1(I, \partial I) \Rightarrow -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_W - M}{D_W + M}\right)\right] = -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_{\mathrm{cont.}} - M}{D_{\mathrm{cont.}} + M}\right)\right]$$

$$= \mathrm{Ind}_{\mathrm{mod-two}} D_{\mathrm{cont.}}$$

But there is no overlap Dirac counterpart.

# Contents

✓ 1. Introduction

We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)

is great but equivalent to the eta invariant of the massive Wilson Dirac op.

✓ 3. K-theory

classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

✓ 4. Massless Dirac ($K^0$ group) vs. massive Dirac ($K^1$ group) in continuum

Counting lines (massive, $K^1$) is easier than counting points (massless, $K^0$).

✓ 5. Main theorem on a lattice

The proof is given by lattice-continuum combined Dirac operator, which is gapped.

✓ 6. Comparison with the overlap Dirac index

we expect wider applications (to APS boundary and real case) than the overlap index.

7. Summary and discussion

# Summary

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than $D_{ov}$ to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978...]

# Summary

$$K^1(I, \partial I)$$

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

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# Backup slides

# What are the weak convergence and strong convergence?

The sequence $v_j$ weakly converges to $v_\infty$

when for arbitrary $w$

$$\lim_{j\to\infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j\to\infty} (v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich's theorem:

$L_1^2$ weak convergence = $L^2$ convergence