The index of lattice Dirac operators and K-theory

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Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), "The index of lattice Dirac operators and K-theory," arXiv:2407.17708, 2501.02873

What is the index of Dirac operators ?

$$D\not{\psi} = 0 \qquad D := \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

we consider
U(1) or SU(N) group

[Atiyah & Singer 1963]

Index theorem

$$\overbrace{n_{+} - n_{-}}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^{2}}\int d^{4}x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) = \boldsymbol{E}\cdot\boldsymbol{B}$$

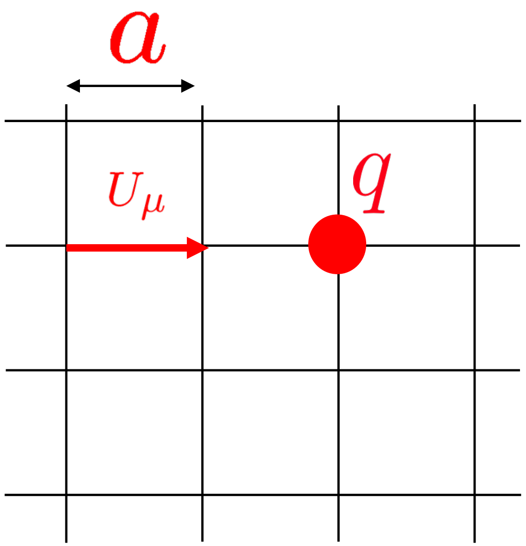
Topological charge

#sol with + chirality

#sol with - chirality

Important both in physics and mathematics to understand gauge field topology, which is non-perturbative.

What is lattice gauge theory?



It is a (non-perturbative) regularization of quantum field theory with lattice spacing a

Gauge fields(gluons) live on links

$$U_{n,\mu} = \exp(igaA_\mu(n+\hat{\mu}/2))$$

Fermions (quarks) live on sites $q_n = q(n)$

The Lagrangian is given by for example,

$$L = \beta \sum_{\mu,\nu=1}^{4} \mathrm{Tr}[U_{n,\mu}U_{n+\mu,\nu}U^\dagger_{n+\nu,\mu}U^\dagger_{n,\nu}] + \bar{q}_n \left[\sum_\mu \gamma_\mu \frac{U_{n,\mu}q_{n+\hat{m}u} - U^\dagger_{n-\hat{m}u,\mu}q_{n-\hat{\mu}}}{2a} + m\right] q_n$$

which converges to QCD Lagrangian in the $a \rightarrow 0$ limit.

Our goal

= A mathematical formulation of the index (theorem) on a lattice.

In continuum, Dirac operator is a differential operator.

$$D\psi = \gamma^\mu(\partial_\mu + iA_\mu)\psi.$$

On lattice, Dirac operator is a difference operator.

$$D^{\mathrm{naive}}\psi = \gamma^\mu[U_\mu(x)\psi(x+\mu a) - U_\mu^\dagger(x-\mu a)\psi(x-\mu a)]/2a.$$

Mathematically nontrivial.

[Related works by mathematicians: Kubota 2020, Yamashita 2021]

Difficulty in lattice gauge theory

Both of Dirac index and topology are difficult on the lattice:

- It is difficult to define the chiral zero modes, since the standard lattice Dirac operators break the chiral symmetry.

- Lattice discretization of space time makes the topology not well-defined.

A traditional solution = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] satisfying the Gingparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$$

a modified chiral symmetry is exact [Luescher 1998],

and the index is well-defined: $\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$

[Hasenfratz et al. 1998]

but this definition is so far limited to even-dimensional flat periodic lattices.

This work = an alternative mathematical formulation of the lattice Dirac index.

In our formulation,

- Chiral symmetry is NOT necessary : the standard Wilson Dirac operator is good enough.
- K theory is used to show the equality to the continuum Dirac index.
- Wider application than the overlap Dirac operator to the systems with nontrivial boundaries and/or mod-two version of the index.

Phys-Math collaborators

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Mathematicians

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Physicist-friendly Dirac index project

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two index version [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]
- Lattice version [Aoki, F, Furuta, Matsuo, Onogi, Yamaguchi 2024, 2025(in preparation)] = this work.

Q. How physicist-friendly?
A. We do not need to take care of chiral symmetry and unphysical boundary conditions in our formulation.

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3. K-theory
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5. Main theorem on a lattice
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7. Summary and discussion

Continuum-> Lattice : derivative -> difference

Continuum Dirac operator

$$D\psi(x) = \gamma^\mu(\partial_\mu)\psi(x) = \int dp\gamma^\mu(ip_\mu)\tilde{\psi}(p)e^{ipx}$$

(A naïve) lattice Dirac operator

$$D\psi(x) = \gamma^\mu \frac{\psi(x+\hat{\mu}a) - \psi(x-\hat{\mu}a)}{2a} = \int dp\gamma^\mu \frac{e^{ip(x+\hat{\mu}a)} - e^{ip(x-\hat{\mu}a)}}{2a}\tilde{\psi}(p)$$

$$= \int dp\gamma^\mu i\frac{\sin p_\mu a}{a}\tilde{\psi}(p)e^{ipx}.$$

a :lattice spacing

$\hat{\mu}$: unit vector in μ direction.

which has zero points at $p_\mu = 0, \quad \frac{\pi}{a}$

(phys) Doublers appear!

(math) Ellipticity is lost!

Wilson Dirac operator

a :lattice spacing

$\hat{\mu}$: unit vector in μ direction.

The Wilson Dirac operator is commonly used in lattice gauge theory.

$$D_W = \sum_\mu \left[\gamma^\mu \frac{\nabla^f_\mu + \nabla^b_\mu}{2} - \frac{a}{2} \nabla^f_\mu \nabla^b_\mu \right]$$

$$\nabla^f \psi(x) = \frac{\psi(x+\hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^b \psi(x) = \frac{\psi(x) - \psi(x-\hat{\mu}a)}{a}$$

The additional term corresponds the Laplacian and the Fourier transformation

$$\sum_\mu \gamma^\mu i \frac{\sin p_\mu a}{a} + \sum_\mu \frac{(1-\cos p_\mu a)}{a}$$

= Large mass term

Except for $p_\mu = 0$

indicates that the doublers cannot excite (recovering ellipticity)due to heavy mass but chiral symmetry (Z_2 grading) is lost: $\gamma_5 D_W + D_W \gamma_5 \neq 0.$

Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:
If $\gamma_5 D + D\gamma_5 = 0,$ we cannot avoid fermion doubling.

(we have to give up Z_2 grading to recover ellipticity)

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

can avoid NN theorem.
But no concrete form was found in ~20 years.

Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \quad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$

$$\gamma_5(1 - aD_{ov}/2)\gamma_5 D_{ov} + \gamma_5 D_{ov}\gamma_5(1 - aD_{ov}/2) = 0.$$

➡ $\Gamma_5 H + H\Gamma_5 = 0.$

$$H = \gamma_5 D_{ov}, \quad \Gamma_5 = \gamma_5\left(1 - \frac{aD_{ov}}{2}\right)$$

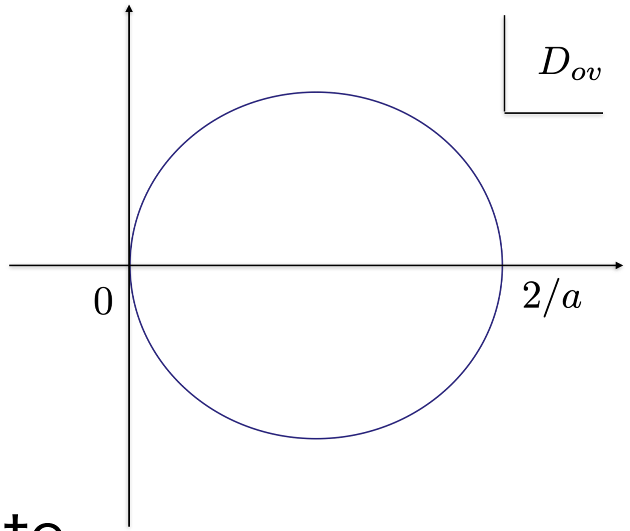
= a modified exact chiral symmetry but $\Gamma_5^2 \neq 1.$

[Luescher 1998]

We can define the index !

[Hasenfratz et al. 1998]

a :lattice spacing



Overlap Dirac spectrum lies on a circle with radius 1/a

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)\psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real 2/a (doubler poles) do not contribute.

$$\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\mathrm{zero\text{-}modes}} \gamma_5 = n_+ - n_-$$

But D_{ov} is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

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$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

What is this ???

η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}}\mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

The Wilson Dirac operator and K-theory

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

$$M = 1/a$$

In this talk, we try to show a deep mathematical meaning of the right-hand side of the equality,

and try to convince you by K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978···] that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology.

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What is fiber bundle (for physicists)?

A united manifold of spacetime (= base manifold) and field (fiber)

$$\phi(x) \to (x, \phi) \in X \times F$$

Spacetime
= base space

Field space
= fiber space

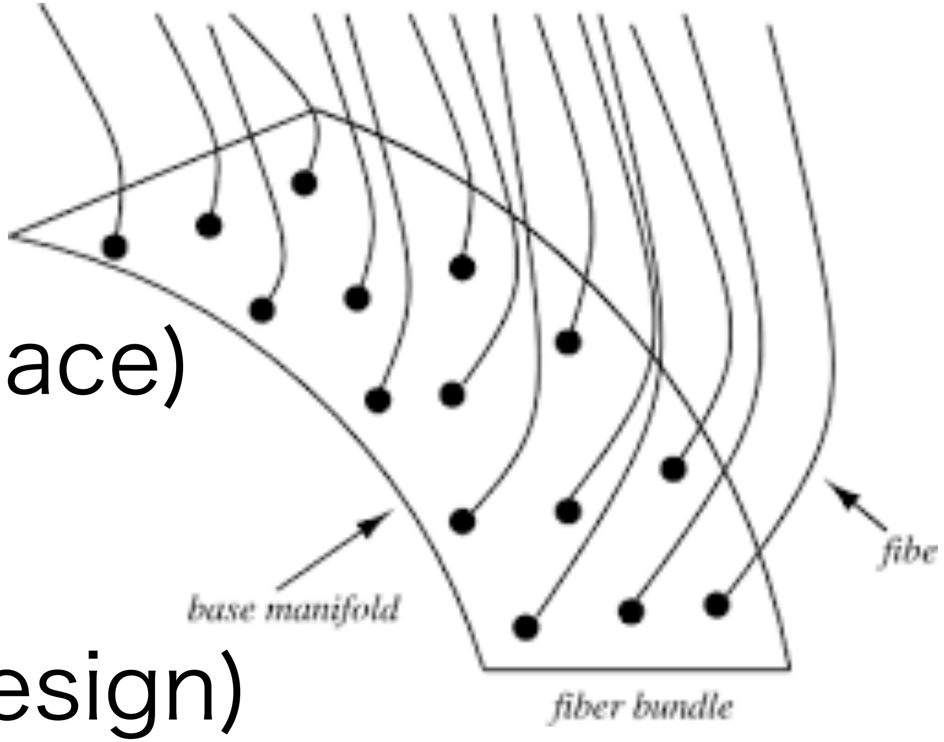
The direct product structure is realized only locally.
In general, it is "twisted" by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by E or $E \to X$

What is fiber bundle? Analogy for (phys) students

X base space (space-time)
  = your head
F fiber (field)
  = your hair
E (= locally XxF) (total space)
  = your hair style
Connection
  = hair wax (local hair design)

Figure from Wolfram Math world



Classification of vector bundles

Let us consider the case with fiber = some vector space.

Compare two vector bundles E_1 and E_2 .

It was proved that the homotopy theory can completely classify the vector bundles. But concrete computation is difficult.

K-theory can classify the vector bundles when their rank is sufficiently large.
(more powerful than the standard (de Rham) cohomology theory).

$K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$
[ ] denotes the equivalence class (concrete definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \to E_2 \qquad D_{12}^\dagger : E_2 \to E_1$$

* To be precise, D acts on the sections of E.

to represent the same element by $[E, D, \gamma]$
where

$$E = E_1 \oplus E_2, \quad D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

* K^0 group describes classification of Dirac operator which anticommutes with chirality operator.

K-theory pushforward

When we are interested in global structure only,
We can forget about details of the base manifold X by taking "one-point compactification" or the K-theory pushforward :

$$G : K^0(X) \to K^0(\mathrm{point})$$

$$[E, D, \gamma] \to [H_E, D, \gamma]$$

The map just forgets all but the chiral symmetry.

H_E : The whole Hilbert space on which D acts.

A lot of information is lost but one (the Dirac operator index) remains.

Suspension isomorphism

The “point” can be suspended to an interval:

There is an isomorphism between

$$K^0(\mathrm{point}) \cong K^1(I, \partial I)$$

$$[H_E, D, \gamma] \leftrightarrow [p^* H_E, D_t]$$

p^* : pull-back of $p : I \to \mathrm{point}.$
we omit in the following.

where the superscript “1” reflects removal of the chirality operator. Instead, the Dirac operator must become one-to-one (no zero mode) at the two endpoints : ∂I

Physical meaning of the isomorphism will be given soon later .

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Atiyah-Singer index

Index theorem

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[H_E, D, \gamma] \in K^0(\mathrm{point})$
But we will show that it is isomorphic to

$$[H_E, \gamma(D+m)] \in K^1(I, \partial I)$$

Eigenvalues of continuum massive Dirac operator

$H(m) = \gamma_5(D_{\text{cont.}} + m)$ on Euclidean even-dimensional manifold.
Gauge group is U(1) or SU(N)

For $D_{\text{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$

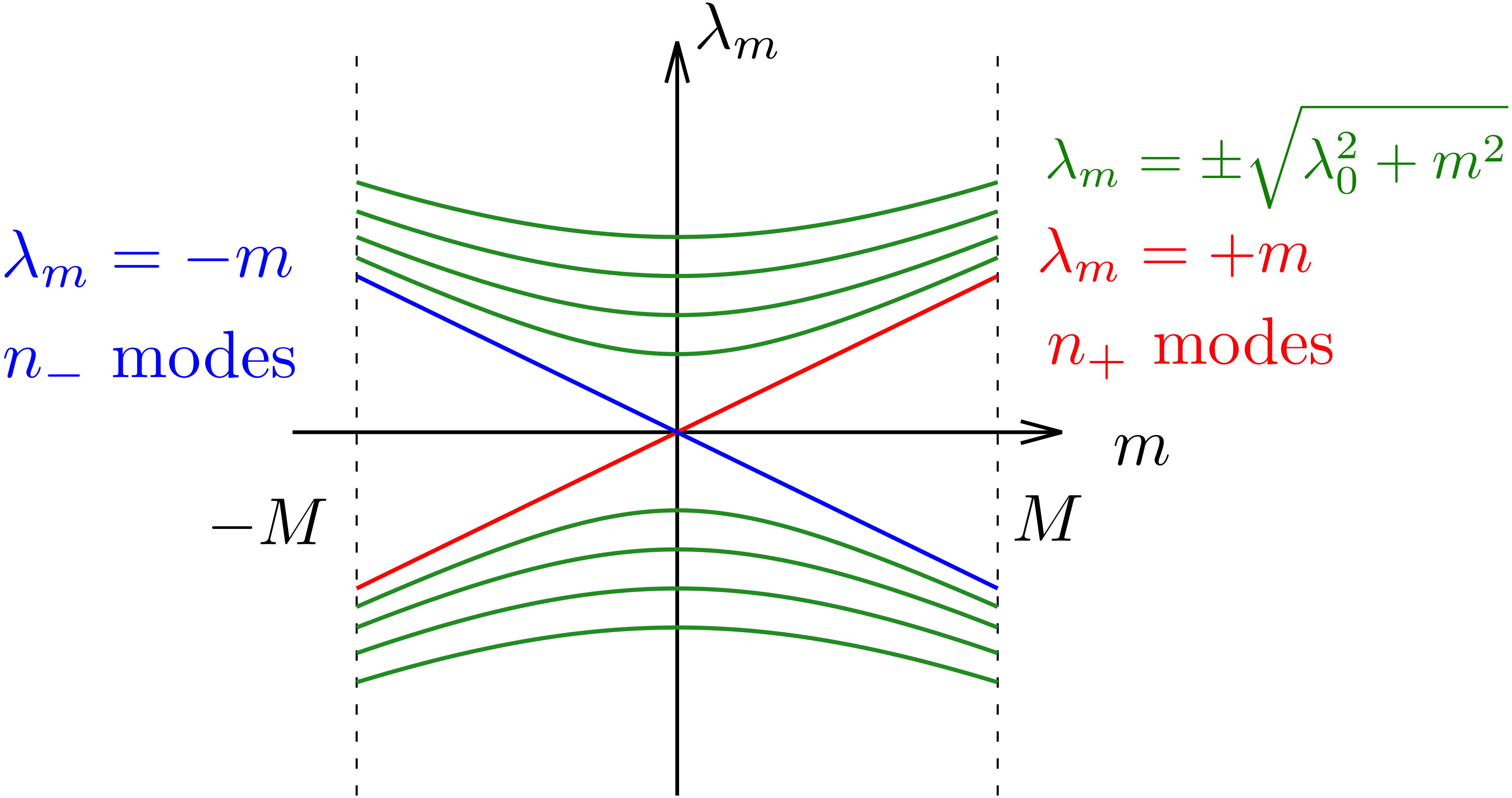
For $D_{\text{cont.}}\phi \neq 0, \quad \{H(m), D_{\text{cont.}}\} = 0.$

The eigenvalues are paired: $H(m)\phi_{\lambda_m} = \lambda_m \phi_{\lambda_m}$

$$H(m)D_{\text{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\text{cont.}}\phi_{\lambda_m}$$

As $H(m)^2 = -D^2_{\text{cont.}} + m^2$, we can write them $\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

Spectrum of $H(m) = \gamma_5(D_{\text{cont.}} + m)$



Spectral flow = Atiyah-Singer index = η invariant

n_+ = # of zero-crossing eigenvalues from - to +

n_- = # of zero-crossing eigenvalues from + to -

$$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$$

$n_+ - n_-$ =: spectral flow of $H(m)$ $\quad m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

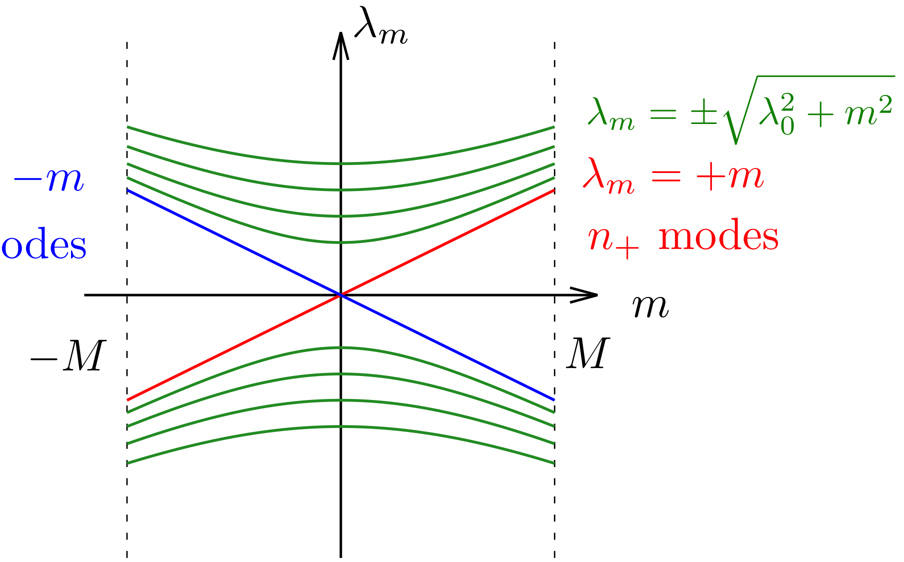
$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

Suspension isomorphism in K theory

Massless:
counting index
by points



Massive:
counting
index by lines

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

point	line=interval
with chirality operator	without chirality operator

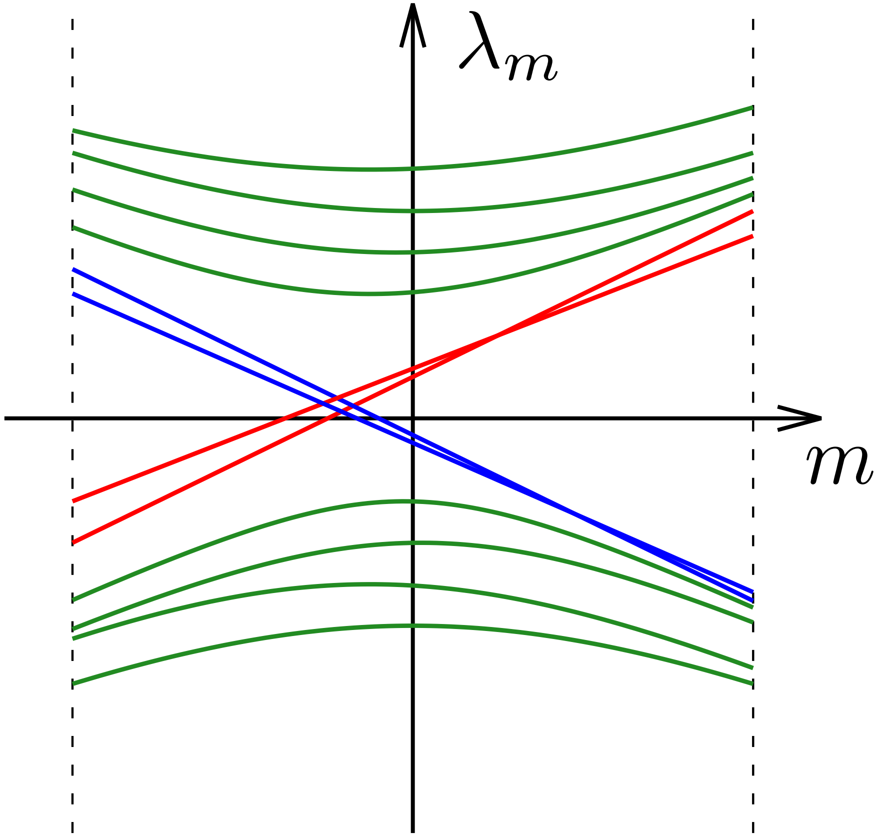
⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard definition:

Where is $m=0$?

What are zero modes?



Eta invariant:

If m= ± M points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

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classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

✓ 4. Massless Dirac (K^0 group) vs. massive Dirac (K^1 group) in continuum

Counting lines (massive, K^1) is easier than counting points (massless, K^0).

5. Main theorem on a lattice

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Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus T^{2n}

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field A_i

D : Dirac operator on sections of E

$$D_{\mathrm{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality (Z_2 grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

Wilson Dirac operator on a lattice

We regularize T^{2n} is by a square lattice with lattice spacing a

(The fiber is still continuous.)

We denote the bundle by E^a and

link variables :

$$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right],$$

$$D_W = \sum_i\left[\gamma^i\frac{\nabla_i^f+\nabla_i^b}{2}-\frac{a}{2}\nabla_i^f\nabla_i^b\right]$$

Wilson term

$$a\nabla_i^f\psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i)-\psi(\boldsymbol{x})$$

$$a\nabla_i^b\psi(\boldsymbol{x}) = \psi(\boldsymbol{x})-U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

Note: In our paper, we consider ”generalized link variables” to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space I = range of mass [-M, M]

boundary ∂I = $\pm$M points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

D_m : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$[(\mathcal{H}, D_m)]$ having the same spectral flow.

Note: K^1 group does NOT require any chirality operator.

Definition of $K^1(I,\partial I)$ group

Group operation: $[(\mathcal{H}^1, D_m^1)] \pm [(\mathcal{H}^2, D_m^2)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D_m^1 & \\ & \pm D_m^2 \end{pmatrix})]$

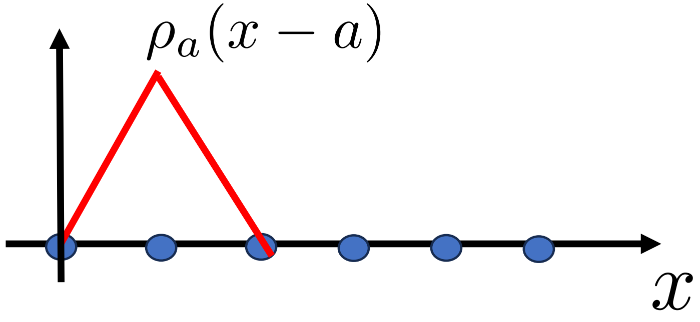
Identity element: $[(\mathcal{H}, D_m)]|_{\text{Spec.flow}=0}$

We compare $[(\mathcal{H}_{\text{cont.}}, \gamma(D_{\text{cont.}} + m))]$ and $[(\mathcal{H}_{\text{lat.}}, \gamma(D_W + m))]$ taking their difference, and confirm if the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & f_a \\ f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has Spectral flow =0 where f_a^* f_a are "mixing mass term" with some "nice" mathematical properties.

$$f_a : H^{\mathrm{lat.}} \to H^{\mathrm{cont.}}$$



maps from finite-dimensional Hilbert space on a discrete lattice to infinite-dimensional continuum one :

$$f_a\phi(x) := a^n \sum_{z\in\mathrm{lattice\ sites}} \rho_a(x-z)U(x,z)\phi(z).$$

$U(x,z)$: parallel transport (or Wilson line) to ensure the gauge invariance.

$\rho_a(x-z)$: weight function (multi-) linearly interpolating the nearest-neighbors.

To control the norm before/after the map, it satisfies

$$\int_{x\in T^n} \rho_a(x-z)d^n x = 1 \qquad a^n \sum_{z\in\mathrm{lattice\ sites}} \rho_a(x-z) = 1.$$

$$f_a^* : H^{\text{cont.}} \to H^{\text{lat.}}$$

Is defined by

$$f_a^* \psi_1(z) := \int_{x \in T^n} \rho_a(z-x) U(x,z)^{-1} \psi_1(x) d^n x.$$

Note) $f_a^* f_a$ is not the identity but smeared around nearest-neighbor sites.
(The gauge invariance is maintained by the Wilson lines.)

Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,

a constant c exists such that

$$||D_i \phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large, D is also large.

This property is nontrivial on a lattice.

$$||\nabla_i^f \phi||^2 \leq c(||\phi||^2 + ||D_W \phi||^2)$$

Without Wilson term, doubler modes would have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important to make the Dirac operator elliptic.

Continuum limit of $f_a^* \quad f_a$

1. For arbitrary $\phi^{\mathrm{lat.}}$

   $\lim_{a\to 0} f_a \phi^{\mathrm{lat.}}$ weakly converges to a $^{\exists}\phi_0^{\mathrm{cont.}} \in L_1^2$

   where L_1^2 is a subspace of $H^{\mathrm{cont.}}$ where the elements and their first derivatives are square integrable.

2. $\lim_{a\to 0} f_a \gamma (D_W + m)\phi^{\mathrm{lat.}}$ weakly converges to $\gamma(D+m)\phi_0^{\mathrm{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a \phi^{\mathrm{lat.}} - \phi^{\mathrm{lat.}}||_{L^2}^2 < ca^2 ||\phi^{\mathrm{lat.}}||_{L_1^2}^2$

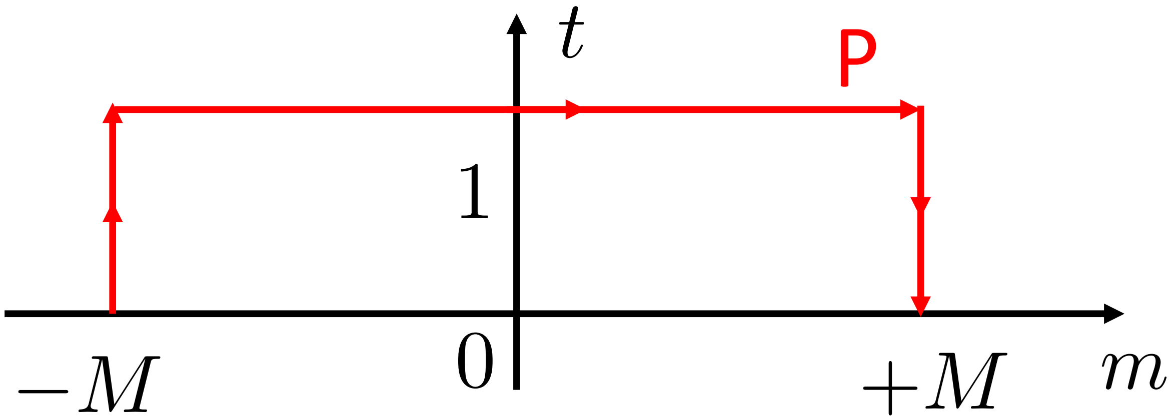
4. For any $\phi^{\mathrm{cont.}} \in L_1^2$, $\lim_{a\to 0} f_a f_a^* \phi_0^{\mathrm{cont.}} = \phi_0^{\mathrm{cont.}}$

Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



Main theorem

There exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\rm cont.} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P
[which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D_{\rm cont.} + m), \quad \gamma(D_W + m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D-M))^{\rm PV\ reg.} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

Proof (by contradiction)

Assume $$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

Continuum limit

Multiplying $\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}}+m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\mathrm{cont.}}+m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$u_\infty, \quad v_\infty$ are

L^2_1 weakly convergent

=

L^2 strongly convergent

(Rellich's theorem)

$$\hat{D}^2_\infty = D^2_{\mathrm{cont.}} + m^2_\infty + t^2_\infty$$

requires

$$m_\infty = t_\infty = 0.$$

Contradiction with $m^2 + t^2 > 0$ along the path P.

Numerical test

We consider a two-dimensional square lattice (or torus)
We set link variables as

$$U_y(x,y) = \exp\left[i\frac{2\pi Q(x-x_0)a}{L_1^2} \right]$$

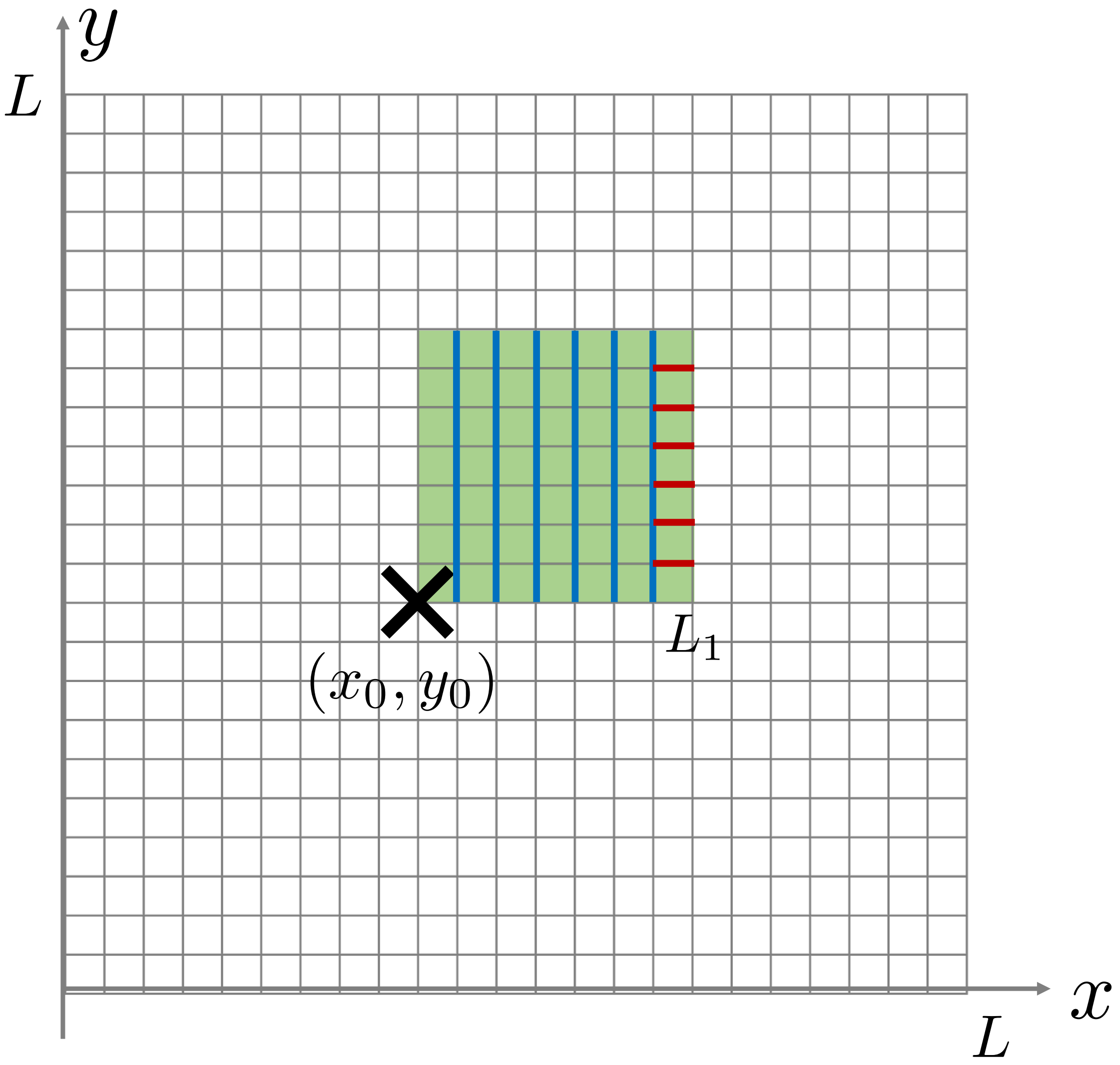
$$U_x(x,y) = \exp\left[-i\frac{2\pi Q(y-y_0)}{L_1} \right]$$

$$\text{others} = 1.$$

Then every green plaquette has a constant curvature

$$U_P(x,y) = \exp\left[i\frac{2\pi Qa^2}{L_1^2} \right]$$

so that geometrical index will be Q.



This constant curvature background can be extended to any even dimensions with SU(N) gauge connections [Cf. Hamanaka-Kajiura 2002].

Massive Wilson Dirac

$$\gamma D_W(m) = \gamma \left[\sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right] + m \right]$$

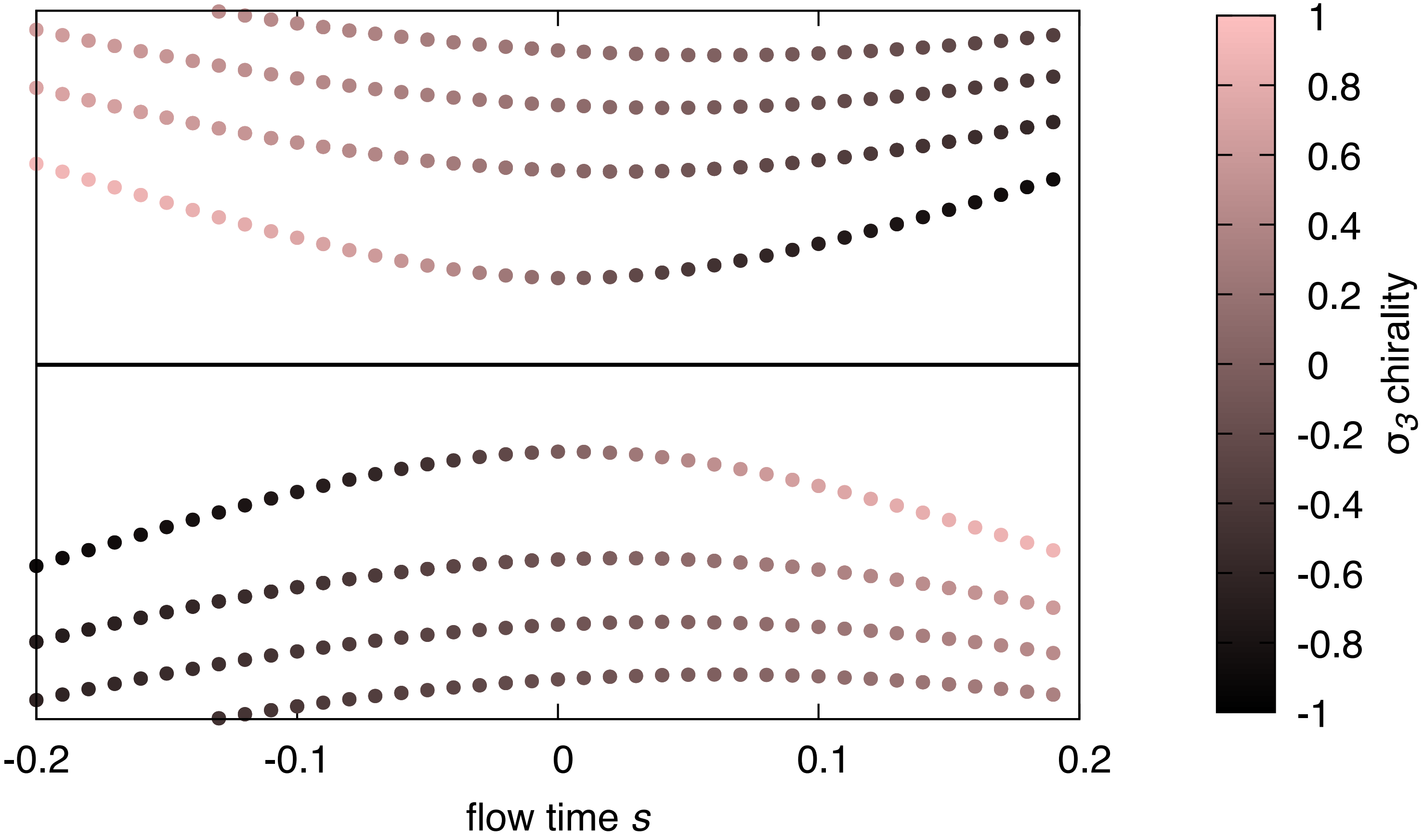
$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x}) \quad a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i)$$

with periodic b.c. in x-direction and anti-periodic b.c. in y direction. We set L=32 and L1=10.

We compute near-zero eigen-spectrum

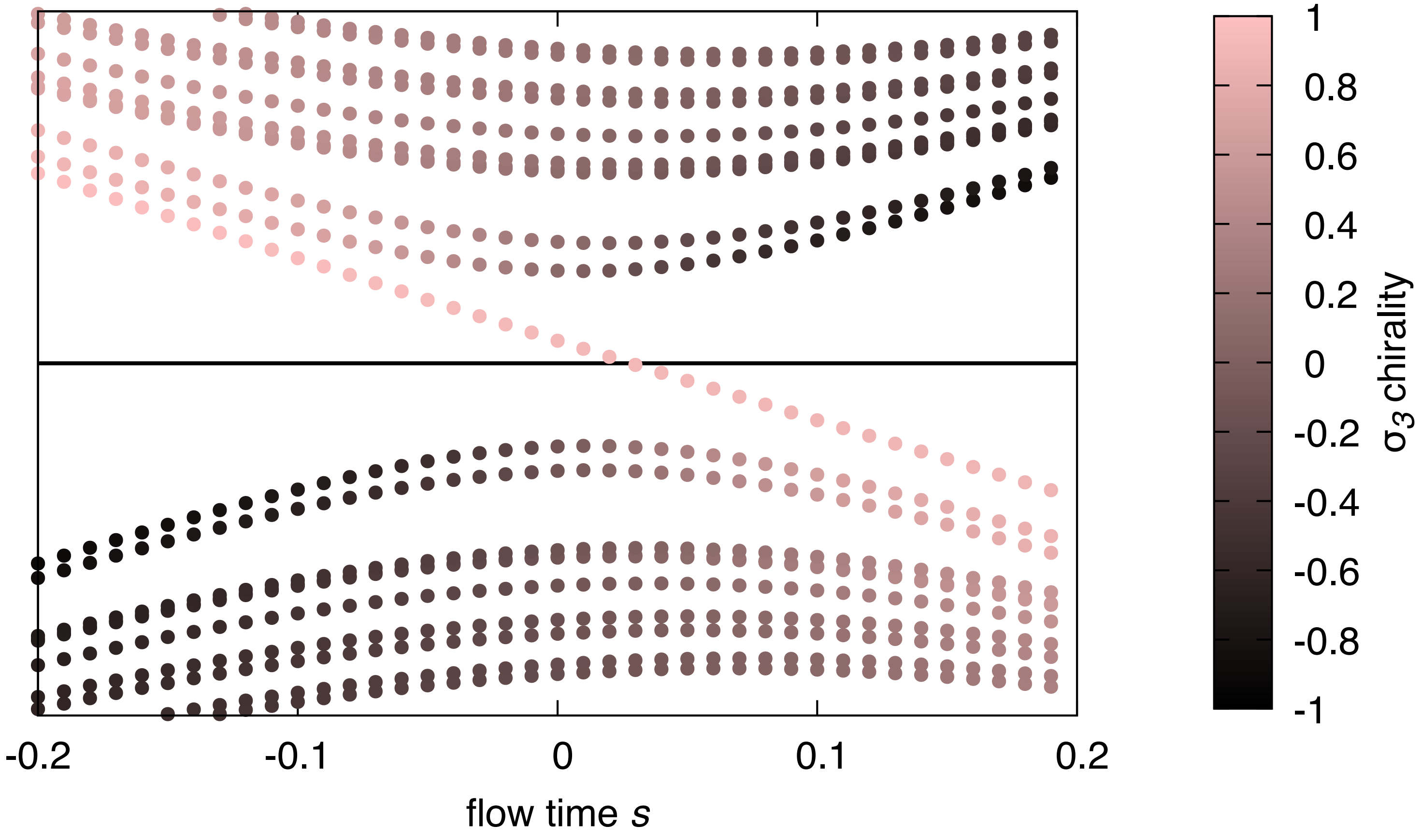
in the range $-1 \le m \le +1$

Wilson Dirac spectrum at Q=0



There is no zero crossing : index=0.

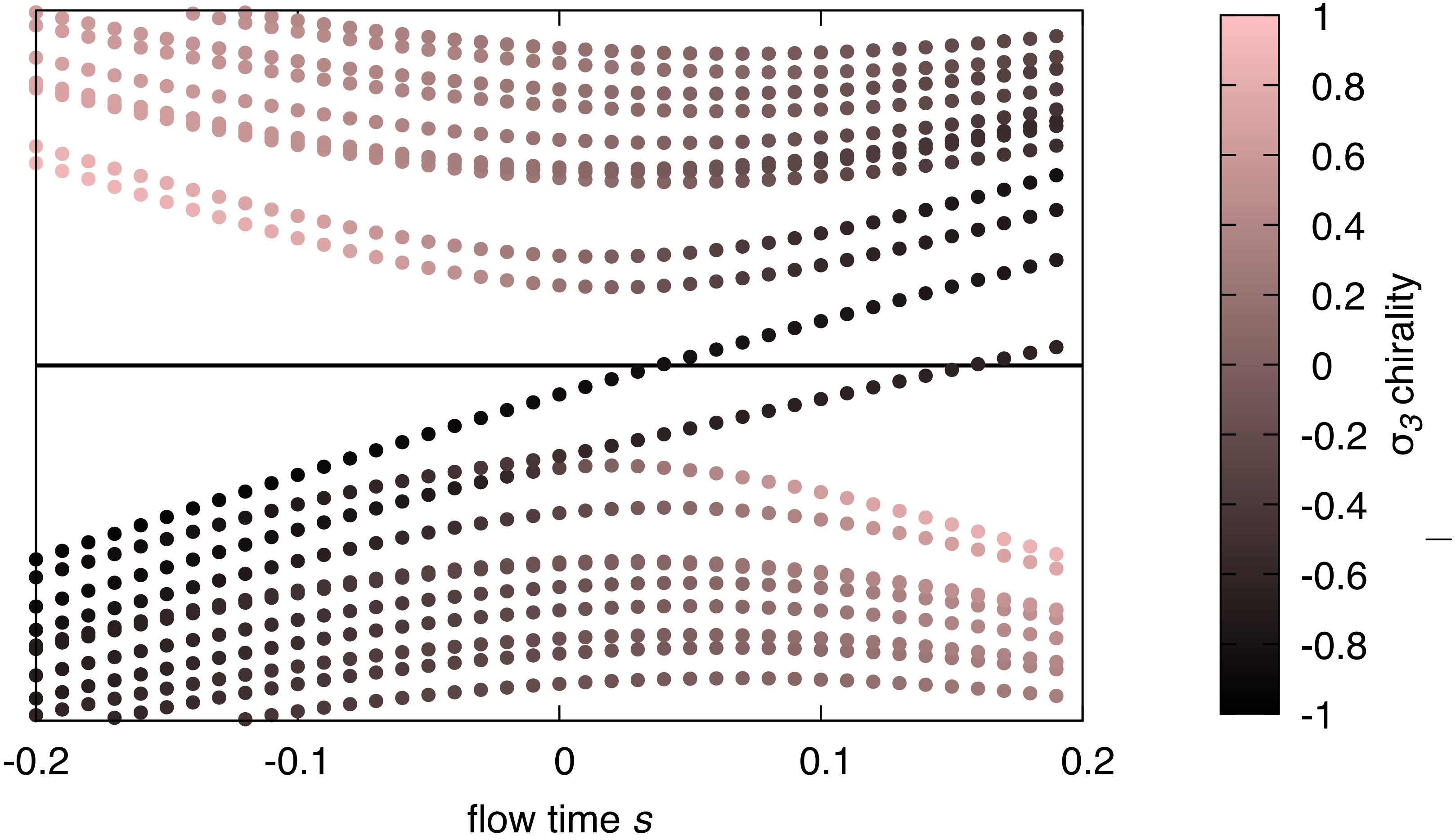
Wilson Dirac spectrum at Q=+1



There is one crossing from positive to negative: index=+1.

$$-\frac{1}{2}\eta(\gamma(D_W - M)) = +1$$

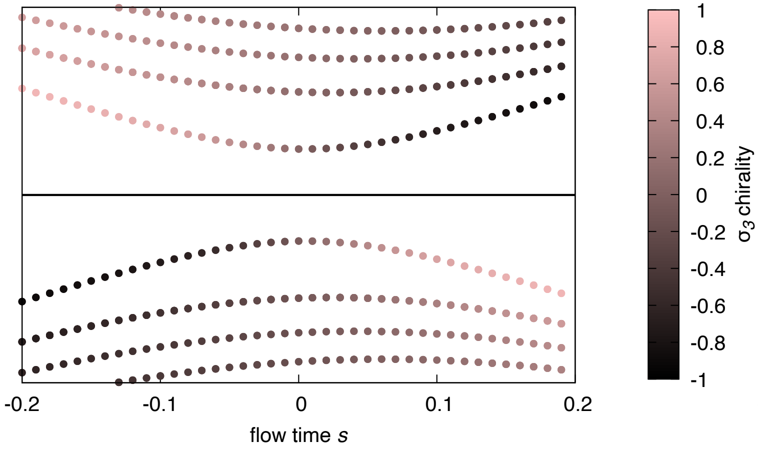
Wilson Dirac spectrum at Q=-2



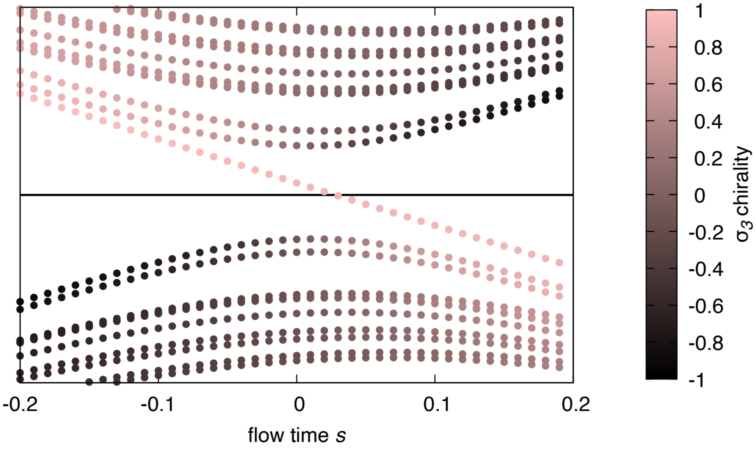
There is two crossings from negative to positive: index=-2.

$$-\frac{1}{2}\eta(\gamma(D_W - M)) = -2$$

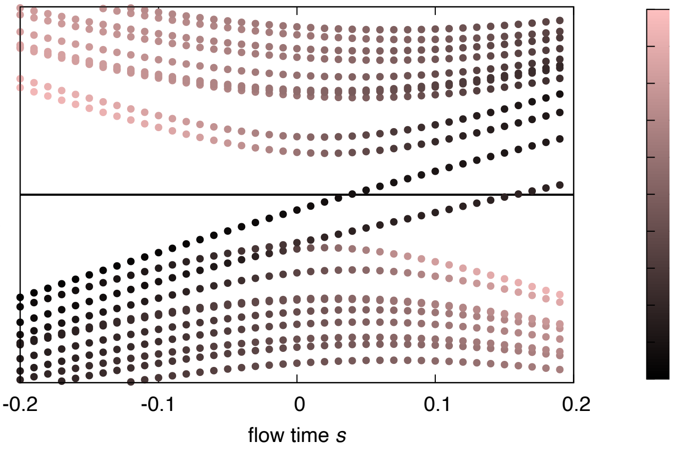
Our 32x32 lattice reproduces the Atiyah-Singer index theorem on a torus.



Index=Q=0



Index=Q=+1



Index=Q=-2

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is great but equivalent to the eta invariant of the massive Wilson Dirac op.

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classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

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6. Applications to a manifold with boundaries and the mod two version

7. Summary and discussion

Wilson Dirac operator is equally good as D_{ov} to describe the index (or may be even better).

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

(so far) limited to even-dimensional flat torus.

By $K^1(I, \partial I)$ for sufficiently small lattice spacings

Suspension isomorphism

K theory knows how to extend the formulation to the systems with (curved) boundaries and/or mod-two version in any dimensions [Aoki,F,Furuta,Matsuo,Onogi, Yamaguchi 2025 (in preparation)].

Application to the manifolds with boundaries

Periodic b.c.

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

Open b.c. (Shamir domain-wall fermion) we can show

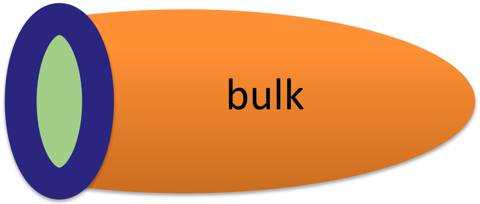
$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) \underset{\Uparrow}{=} -\frac{1}{2}\eta(\gamma_5(D^{\mathrm{cont.}}_{DW})) \underset{\Uparrow}{=} \mathrm{Ind}_{\mathrm{APS}}D^{\mathrm{cont.}}$$

[perturbative equality by F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019 Mathematical proof on-going].

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

But the overlap Dirac op. is missing because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

Atiyah-Patodi-Singer index theorem [1975]



boundary

bulk

curvature

$$Ind(D_{\rm APS}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

* example of 4-dimensional
flat Euclidean space with boundary at $x_4=0$.

Numerical test on a 2D disk

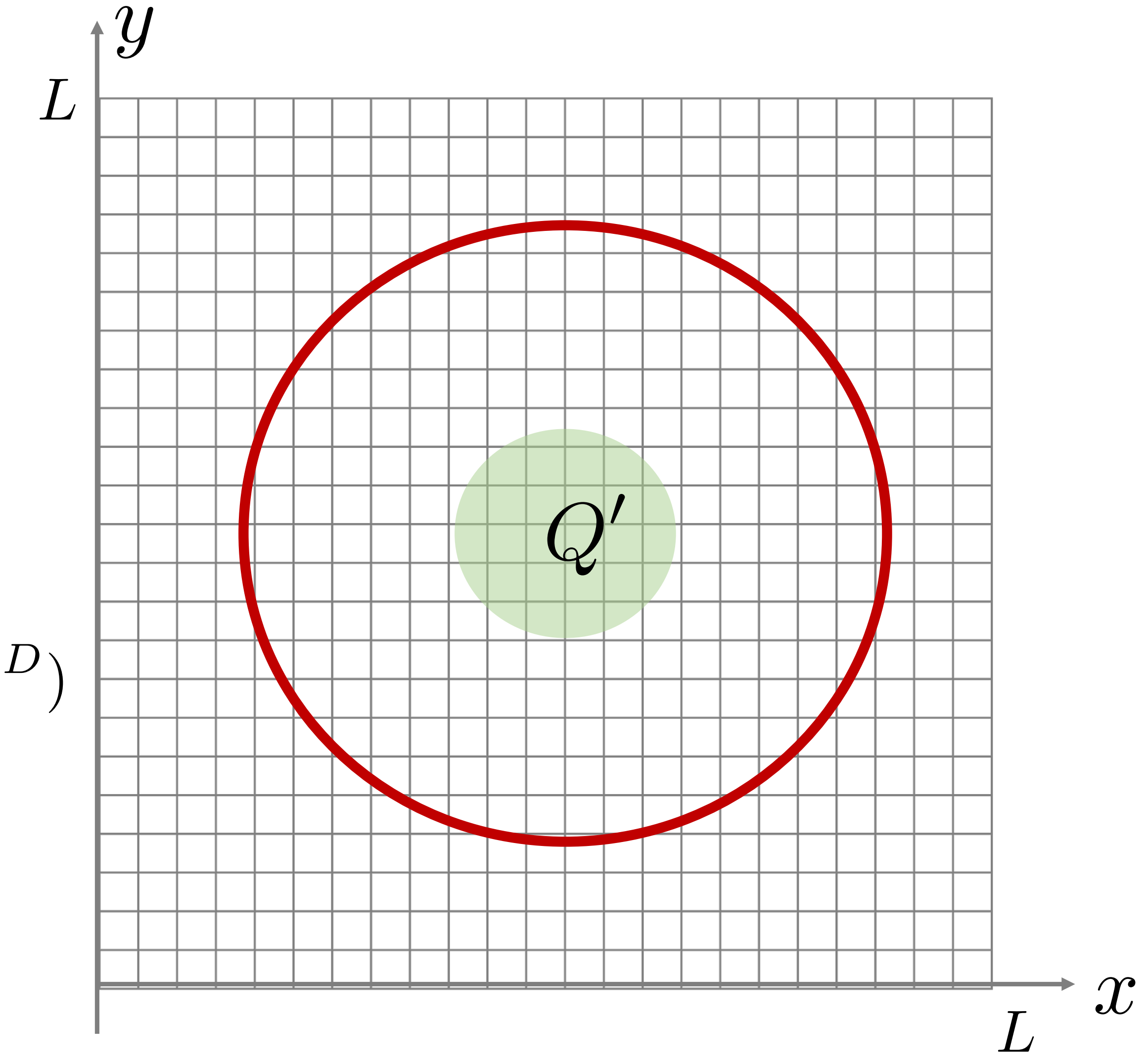
We put a circular curved domain-wall : m=-s inside, m=+1 outside and change s from -1 to 1.

We put U(1) flux Q’ and numerically check if the APS index theorem

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

holds or not.

L=32, DW radius=10, flux radius=6.

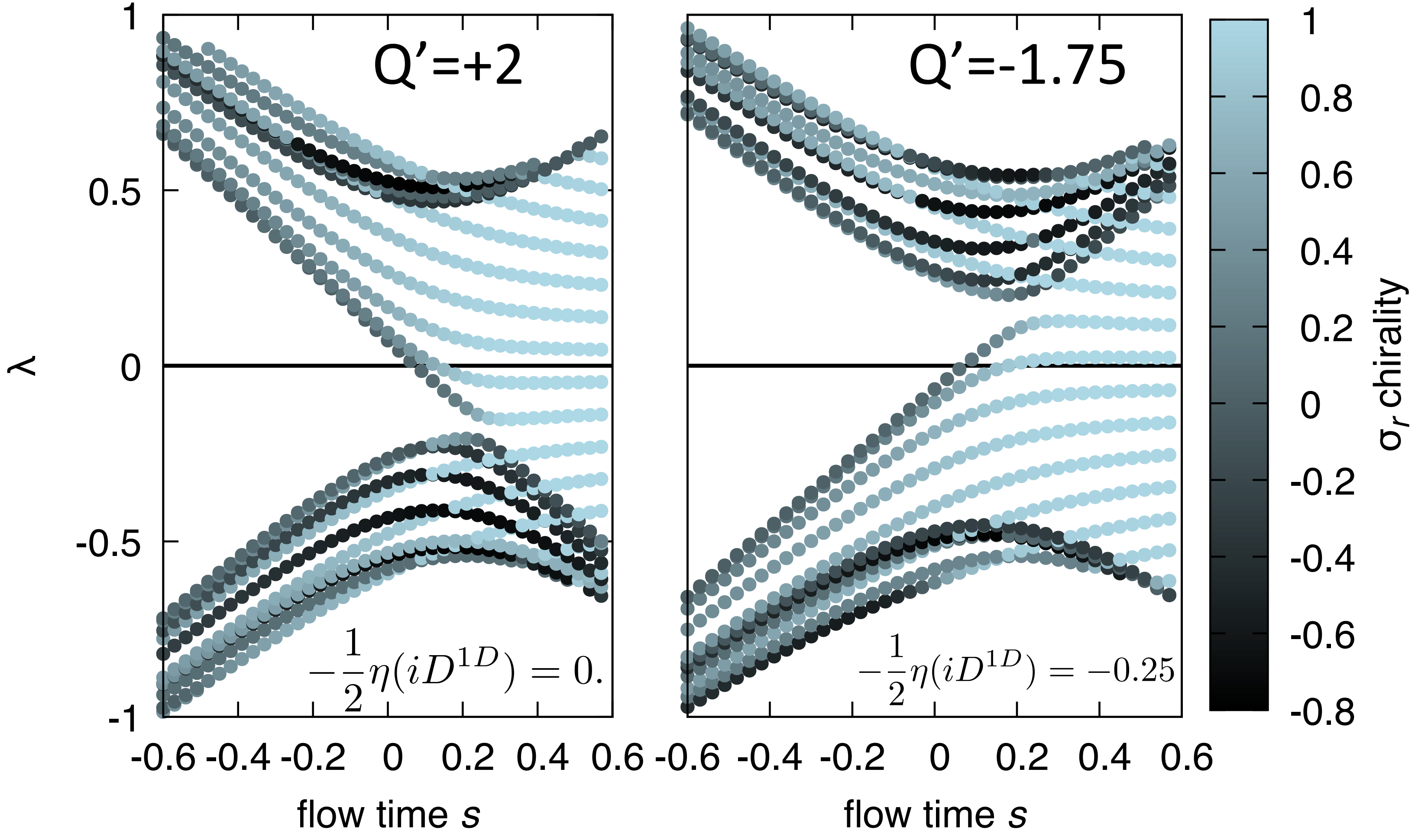


Dirac spectrum on a 2D disk

(Preliminary results)

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$



Edge-localized modes appear on the 1-dimensional circle domain-wall.

The estimated eta invariant is consistent with the APS index.

Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I,\partial I) \Rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D-M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we obtain the mod-2 spectral flow:

$$KO^0(I,\partial I) \Rightarrow -\frac{1}{2}\left[1-\mathrm{sgn}\det\left(\frac{D_W-M}{D_W+M}\right)\right] = -\frac{1}{2}\left[1-\mathrm{sgn}\det\left(\frac{D_{\mathrm{cont.}}-M}{D_{\mathrm{cont.}}+M}\right)\right]$$

$$= \mathrm{Ind}_{\mathrm{mod-two}} D_{\mathrm{cont.}}$$

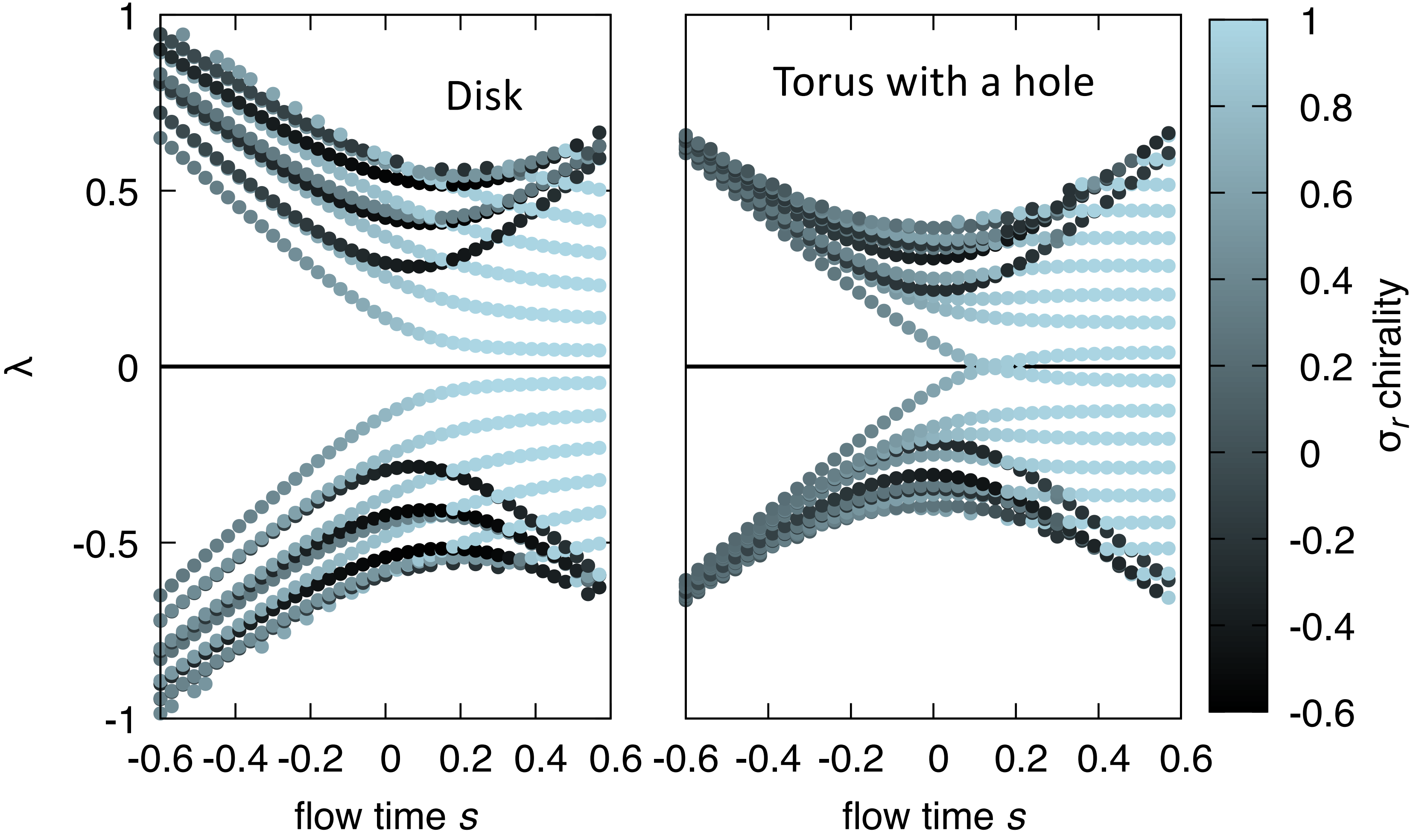
[F, Furuta, Matsuki, Matuso, Onogi, Yamaguchi, Yamashita 2020].

But there is no overlap Dirac counterpart.

Dirac spectrum on a 2D disk

$$H_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 m(s,r)$$

(Preliminary results)



Free (Majorana) Wilson fermion Dirac spectral flow agrees with the mod-two APS index.

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Our K-theoretic formulation has a wider application than the overlap index.

7. Summary and discussion

Summary

$K^1(I,\partial I)$

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deep mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory:

It gives a united formulation of AS, APS, and/or mod-2 version indices.

Backup slides

What are the weak convergence and strong convergence?

The sequence v_j weakly converges to v_∞

when for arbitrary w

$$\lim_{j\to\infty}\langle (v_j - v_\infty), w\rangle = 0.$$

Note) $\lim_{j\to\infty}(v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich’s theorem:

L_1^2 weak convergence = L^2 convergence