# The index of lattice Dirac operators and K-theory

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# K-theory for lattice gauge theory

We discuss topology of lattice gauge theory using K-theory.

K-theory in condensed matter physics is often used in momentum space.

But in this talk, we discuss it in a discrete position space.

- we do not assume translational invariance -

We believe our work is nontrivial both in physics and mathematics.

# What is the index of Dirac operators ?

$$D\psi = 0 \quad D := \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

we consider
U(1) or SU(N) group

[Atiyah & Singer 1963]

Index theorem

$$\underbrace{n_{+} - n_{-}}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) = \boldsymbol{E}\cdot\boldsymbol{B}$$

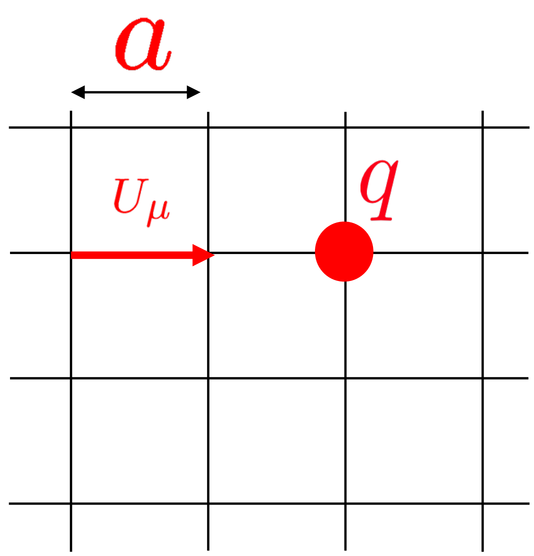
#sol with + chirality

#sol with - chirality

Topological charge = winding number

Important both in physics and mathematics to understand gauge field topology, which is non-perturbative.

# What is lattice gauge theory?




It is a (non-perturbative) regularization of QFT with lattice spacing $a$

Gauge fields(gluons) live on links

$$U_{n,\mu} = \exp(igaA_\mu(n + \hat{\mu}/2))$$

Fermions (quarks) live on sites

$$q_n = q(n)$$

On the lattice, path-integrals = finite-dimensional mathematically well-defined integrals

# Our goal

= A mathematical formulation of the index (theorem) on a lattice.

In continuum, Dirac operator is a differential operator.

$$D\psi = \gamma^{\mu}(\partial_{\mu} + iA_{\mu})\psi.$$

On lattice, Dirac operator is a difference operator.

$$D^{\mathrm{naive}}\psi = \gamma^{\mu}[U_{\mu}(x)\psi(x+\mu a) - U_{\mu}^{\dagger}(x-\mu a)\psi(x-\mu a)]/2a.$$

Mathematically nontrivial.

[Related works by mathematicians: Kubota 2020, Yamashita 2021]

# Difficulty in lattice gauge theory

Both of Dirac index and topology are difficult on the lattice:

- It is difficult to define the chiral zero modes, since the standard lattice Dirac operators break the chiral symmetry.
- Lattice discretization of space time makes the topology not well-defined.

# A traditional solution = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] satisfying the Gingparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$$

a modified chiral symmetry is exact [Luescher 1998],

and the index is well-defined: $$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

[Hasenfratz et al. 1998]

but this definition is so far limited to even-dimensional periodic square lattices (whose continuum limit is a flat torus).

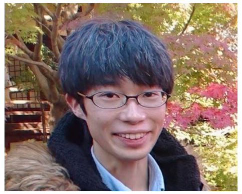
## This work = an alternative mathematical formulation of the lattice Dirac index.

In our formulation,

- Chiral symmetry is NOT necessary : massive Wilson Dirac operator is good enough.
- K theory is used to show the equality to the continuum Dirac index.
- Wider application than the overlap Dirac operator to the systems with (curved) boundaries and/or mod-two version of the index.

# Phys-Math collaborators

## Physicists



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# Physicist-friendly Dirac index project

(no need for chiral symmetry and boundary conditions)

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly index on general curved manifolds [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two index version [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]

} continuum studies

- Lattice perturbative test (on flat torus) [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]
- General lattice version [Aoki, F, Furuta, Matsuo, Onogi, Yamaguchi 2024, 2025] = this work.

# Contents

# Continuum derivative -> Lattice difference

Continuum Dirac operator

$$D\psi(x) = \gamma^\mu(\partial_\mu)\psi(x) = \int dp\gamma^\mu(ip_\mu)\tilde{\psi}(p)e^{ipx}$$

(A naïve) lattice Dirac operator

$$D\psi(x) = \gamma^\mu \frac{\psi(x+\hat{\mu}a) - \psi(x-\hat{\mu}a)}{2a} = \int dp\gamma^\mu \frac{e^{ip(x+\hat{\mu}a)} - e^{ip(x-\hat{\mu}a)}}{2a}\tilde{\psi}(p)$$

$$= \int dp\gamma^\mu i\frac{\sin p_\mu a}{a}\tilde{\psi}(p)e^{ipx}.$$

$a$ :lattice spacing

$\hat{\mu}$ : unit vector in μ direction.

which has zero points at

$$p_\mu = 0, \quad \frac{\pi}{a}$$

(phys) Doublers appear!

(math) Ellipticity [uniqueness of zero points] is lost!

# Wilson Dirac operator

$a$ :lattice spacing

$\hat{\mu}$ : unit vector in μ direction.

The Wilson Dirac operator is commonly used in lattice gauge theory.

$$D_W = \sum_\mu \left[ \gamma^\mu \frac{\nabla^f_\mu + \nabla^b_\mu}{2} - \frac{a}{2} \nabla^f_\mu \nabla^b_\mu \right]$$

$$\nabla^f \psi(x) = \frac{\psi(x+\hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^b \psi(x) = \frac{\psi(x) - \psi(x-\hat{\mu}a)}{a}$$

The additional term corresponds the Laplacian and the Fourier transformation

$$\sum_\mu \gamma^\mu i \frac{\sin p_\mu a}{a} + \sum_\mu \frac{(1-\cos p_\mu a)}{a}$$

= Large mass term Except for $p_\mu = 0$

indicates that the doublers cannot excite (recovering ellipticity) due to heavy mass. But chiral symmetry ($Z_2$ grading) is lost instead:

$$\gamma_5 D_W + D_W \gamma_5 \neq 0.$$

# Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:
If $\gamma_5 D + D\gamma_5 = 0,$ we cannot avoid fermion doubling.

(we have to give up $Z_2$ grading to recover ellipticity)

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

can avoid NN theorem.
But no concrete form was found in ~20 years.

# Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \quad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$

$$\gamma_5(1 - aD_{ov}/2)\gamma_5 D_{ov} + \gamma_5 D_{ov}\gamma_5(1 - aD_{ov}/2) = 0.$$

⇒ $\Gamma_5 H + H\Gamma_5 = 0.$

$$H = \gamma_5 D_{ov}, \quad \Gamma_5 = \gamma_5\left(1 - \frac{aD_{ov}}{2}\right)$$

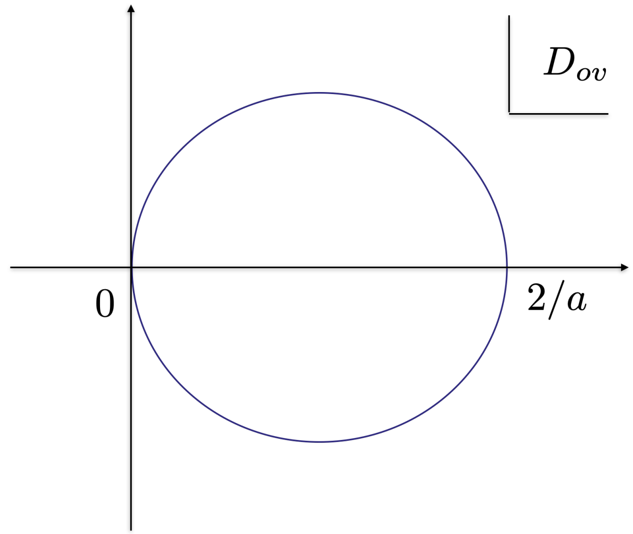
= a modified exact chiral symmetry (but $\Gamma_5^2 \neq 1$.)

[Luescher 1998]

# We can define the index !

[Hasenfratz et al. 1998]

$a$ :lattice spacing




Overlap Dirac spectrum lies on a circle with radius 1/a

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) \psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real 2/a (doubler poles) do not contribute.

$$\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\text{zero-modes}} \gamma_5 = n_+ - n_-$$

But $D_{ov}$ is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

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$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

What is this ???

# η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}} \mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

# The Wilson Dirac operator and K-theory

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

In this talk, we try to show
a deep mathematical meaning of the right-hand side,
and try to convince you by K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…]
that the massive Wilson Dirac operator is an equally good or even better object than $D_{ov}$ to describe the gauge field topology.

# Contents

# What is fiber bundle (for physicists)?

A united manifold of spacetime (= base manifold) and field (fiber)

$$\phi(x) \to (x, \phi) \in X \times F$$

Spacetime
= base space

Field space
= fiber space

The direct product structure is realized only locally.
In general, it is "twisted" by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by $E$ or $E \to X$

# What is fiber bundle? Analogy for (phys) students

X base space (space-time)

= your head

F fiber (field)

= your hair

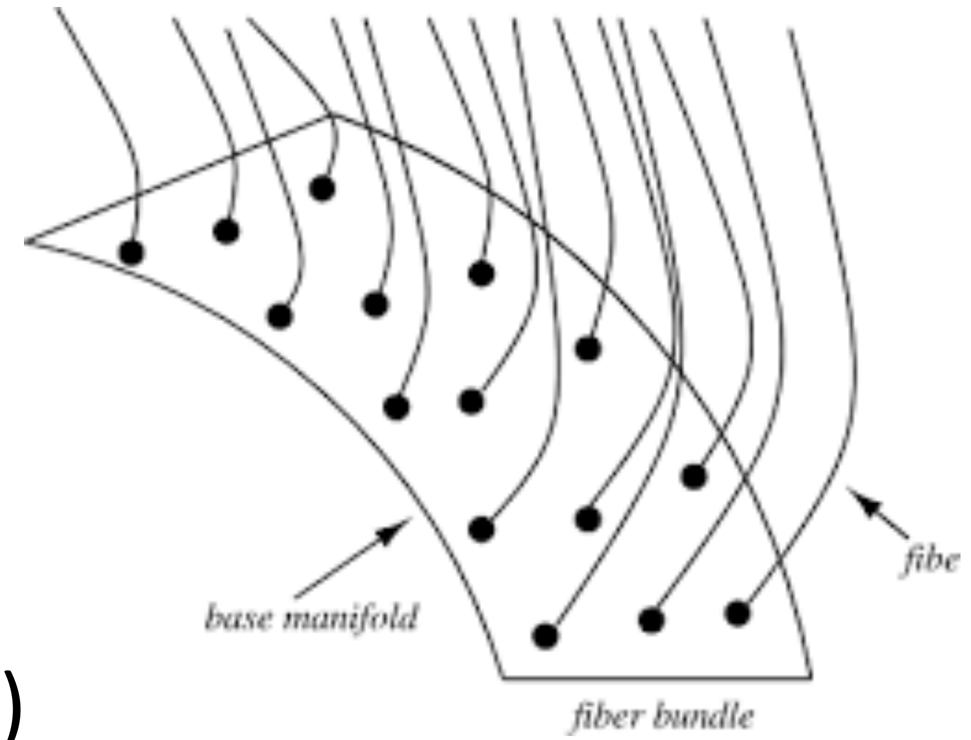
E (= locally XxF) (total space)

= your hair style

Connection

= hair wax (local hair design)

Figure from Wolfram Math world



# Classification of vector bundles

Let us consider the case with fiber = some vector space.

Compare two vector bundles $E_1$ and $E_2$.

It was proved that the homotopy theory can completely classify the vector bundles. But concrete computation is difficult.

K-theory can classify the vector bundles when their ranks are sufficiently large. (more powerful than the standard (de Rham) cohomology theory).

# $K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$

[ ] denotes the equivalence class (concrete definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12}: E_1 \to E_2 \qquad D_{12}^\dagger : E_2 \to E_1$$

* To be precise, D acts on the sections of E.

to represent the same element by $[E, D, \gamma]$

where

$$E = E_1 \oplus E_2, \quad D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

* $K^0$ group describes classification of Dirac operator which anticommutes with chirality operator.

# K-theory pushforward

When we are interested in global structure only,
We can forget about details of the base manifold X by taking
or the K-theory pushforward ( ~ "integration over X" ) :

$$G : K^0(X) \to K^0(\text{point})$$
$$[E, D, \gamma] \to [H_E, D, \gamma]$$

The map just forgets X.

$$H_E : \text{The whole Hilbert space on which } D \text{ acts.}$$

A lot of information is lost but one (the Dirac operator index) remains.

# Suspension isomorphism

The "point" can be suspended to an interval:

There is an isomorphism between

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

One-parameter deformation of Dirac operator

$$[H_E, D, \gamma] \leftrightarrow [p^* H_E, D_t]$$

$p^*$ : pull-back of $p : I \to$ point.
we omit in the following.

where the superscript "1" reflects removal of the chirality operator.

* The Dirac operator must become one-to-one (no zero mode) at the two endpoints : $\partial I$

Physical meaning of the isomorphism will be given soon later .

# Contents

# Atiyah-Singer index

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality    #sol with - chirality

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[H_E, D, \gamma] \in K^0(\mathrm{point})$

But we will show that it is isomorphic to

$$[H_E, \gamma(D+m)] \in K^1(I, \partial I) \qquad m \in [-M, M] =: I$$

# Eigenvalues of continuum massive Dirac operator

$H(m) = \gamma_5 (D_{\text{cont.}} + m)$ On a Euclidean even-dimensional manifold.

For $D_{\text{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$

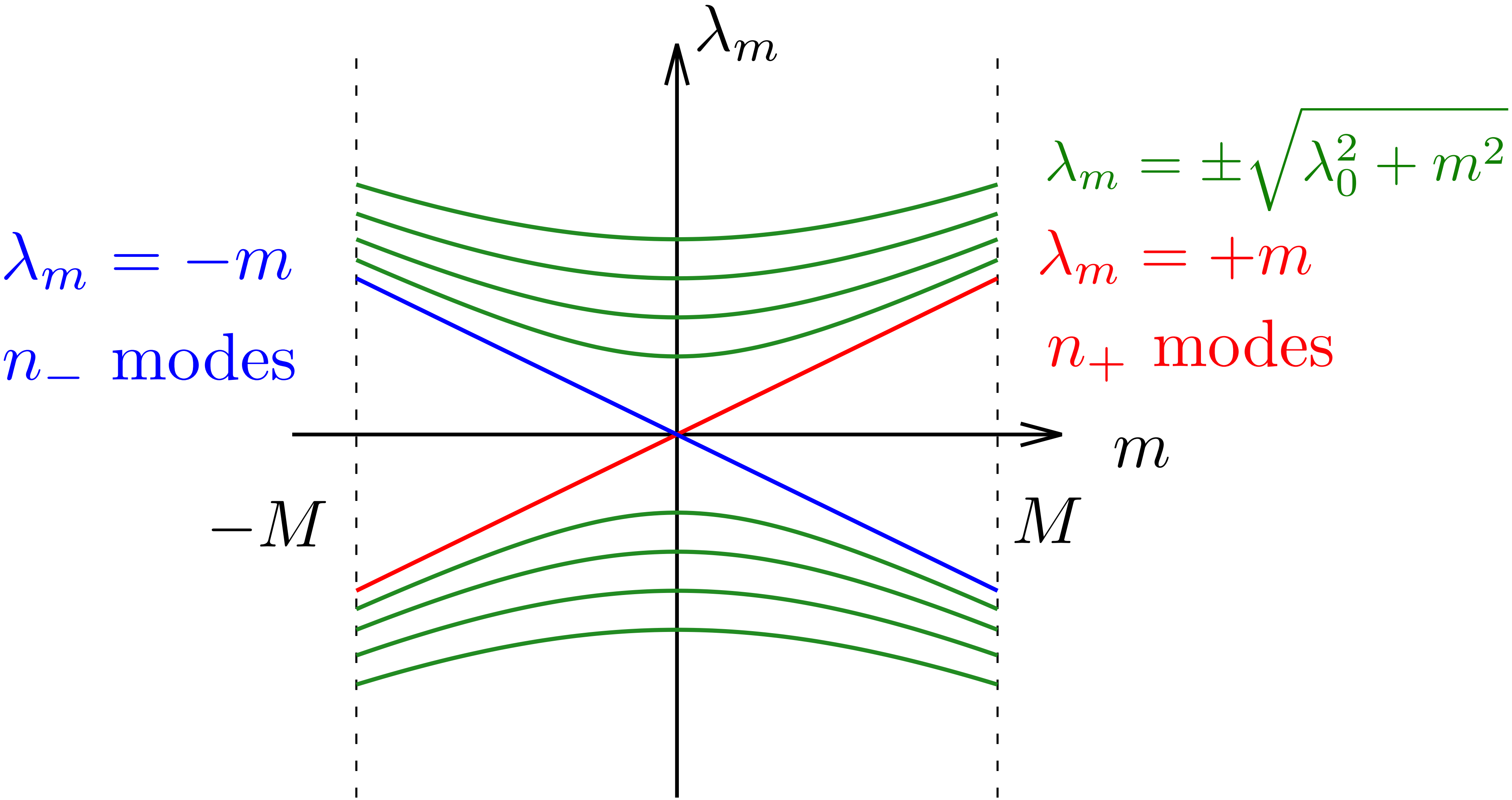
For $D_{\text{cont.}}\phi \neq 0, \quad \{H(m), D_{\text{cont.}}\} = 0.$

The eigenvalues are paired: $H(m)\phi_{\lambda_m} = \lambda_m \phi_{\lambda_m}$

$$H(m) D_{\text{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\text{cont.}}\phi_{\lambda_m}$$

As $H(m)^2 = -D_{\text{cont.}}^2 + m^2$ , we can write them $\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

Spectrum of $H(m) = \gamma_5(D_{\text{cont.}} + m)$



# Spectral flow = Atiyah-Singer index = η invariant

$n_+$ = # of zero-crossing eigenvalues from - to +

$n_-$ = # of zero-crossing eigenvalues from + to -

$$H(m) = \gamma_5(D_{\rm cont.} + m)$$

$n_+ - n_-$ =: spectral flow of $H(m)$ $m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

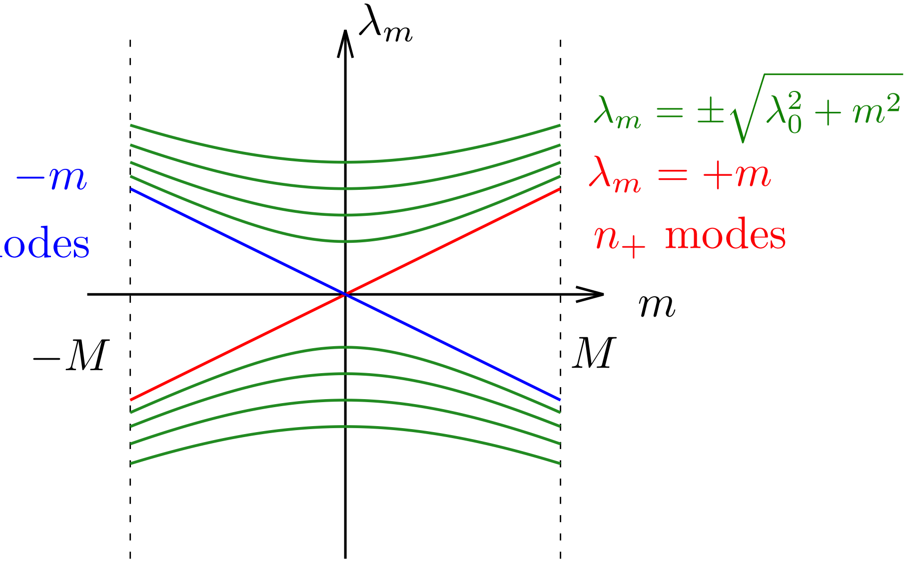
$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

# Suspension isomorphism in K theory

Massless: counting index by points




Massive: counting index by lines

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

point — with chirality operator

line=interval — without chirality operator

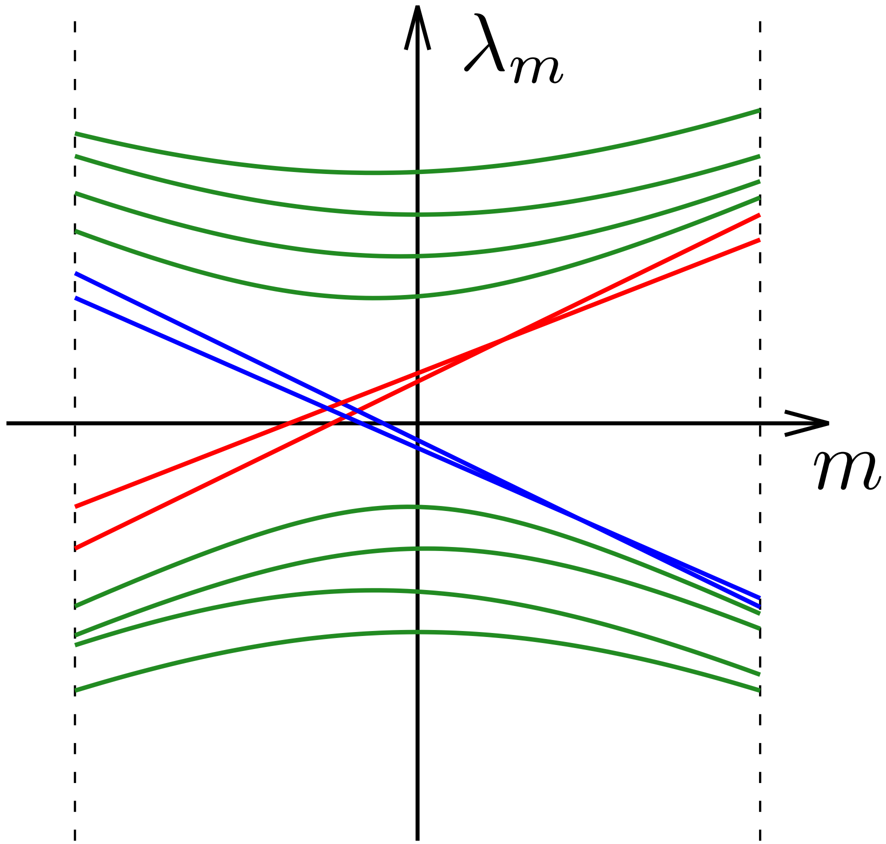
⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard definition:

Where is m=0?

What are zero modes?




Eta invariant:

If m= ± M points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

# Contents

# Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus $T^{2n}$

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field $A_i$

D : Dirac operator on sections of E

$$D_{\mathrm{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality ($Z_2$ grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

# Lattice link variables

We regularize $T^{2n}$ is by a square lattice with lattice spacing $a$

(The fiber is still continuous.)

We denote the bundle by $E^a$ and

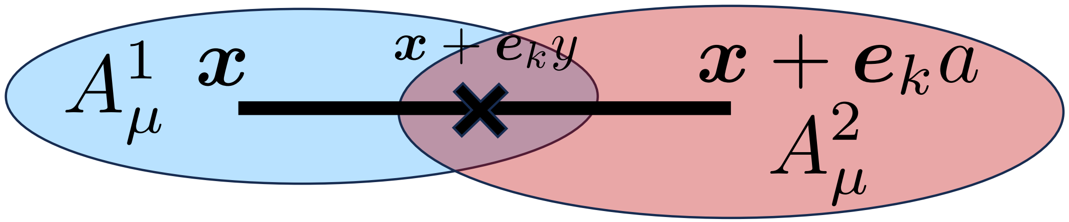
link variables : $$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}' + \boldsymbol{e}_k l)dl\right]$$

Note: In our paper, we consider "generalized link variables" to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

*) When a patch-overlap is on the way of the Wilson line,

$$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^y A^1_k(\boldsymbol{x} + \boldsymbol{e}_k l)dl\right] g_{12}(\boldsymbol{x} + \boldsymbol{e}_k y) P\exp\left[i\int_y^a A^2_k(\boldsymbol{x} + \boldsymbol{e}_k l)dl\right]$$

Transition function




We can show

$$\frac{\partial}{\partial y} U_k(\boldsymbol{x}) = 0.$$

# Wilson Dirac operator on a lattice

$$D_W = \sum_i \left[ \gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right]$$

Wilson term

$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i)$$

* In mathematics, the Wilson term is important in that it guarantees the ellipticity.

# Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space $I$ = range of mass [-M, M]

boundary $\partial I$ = $\pm$M points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

$D_m$ : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$$[(\mathcal{H}, D_m)]$$ having the same spectral flow.

Note: $K^1$ group does NOT require any chirality operator and

does NOT distinguish the continuum and lattice Hilbert spaces.

# Definition of $K^1(I, \partial I)$ group

Group operation: $[(\mathcal{H}^1, D^1_m)] \pm [(\mathcal{H}^2, D^2_m)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D^1_m & \\ & \pm D^2_m \end{pmatrix})]$

Identity element: $[(\mathcal{H}, D_m)]|_{\mathrm{Spec.flow}=0}$

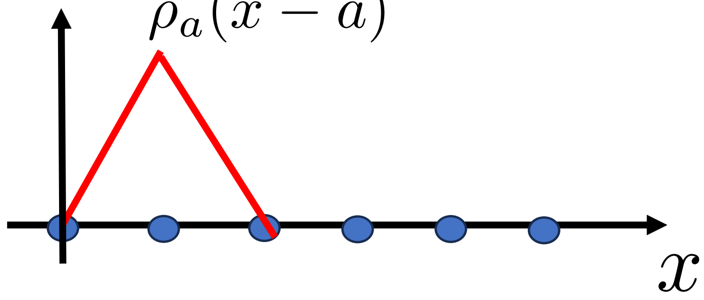
We compare $[(\mathcal{H}_{\mathrm{cont.}}, \gamma(D_{\mathrm{cont.}} + m))]$ and $[(\mathcal{H}_{\mathrm{lat.}}, \gamma(D_W + m))]$

taking their difference, and confirm if the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & f_a \\ f^*_a & -\gamma(D_W + m) \end{pmatrix}$$

has Spectral flow =0 where $f^*_a$ $f_a$ are "mixing mass term" with some "nice" mathematical properties.

$$f_a : H^{\mathrm{lat.}} \to H^{\mathrm{cont.}}$$




maps from finite-dimensional Hilbert space on a discrete lattice to infinite-dimensional continuum one :

$$f_a\phi(x) := a^n \sum_{z\in\mathrm{lattice\ sites}} \rho_a(x-z)U(x,z)\phi(z).$$

$U(x,z)$ : parallel transport (or Wilson line) to ensure the gauge invariance.

$\rho_a(x-z)$: weight function (multi-) linearly interpolating the nearest-neighbors.

To control the norm before/after the map, it satisfies

$$\int_{x\in T^n} \rho_a(x-z)d^n x = 1 \qquad a^n \sum_{z\in\mathrm{lattice\ sites}} \rho_a(x-z) = 1.$$

$$f_a^* : H^{\text{cont.}} \to H^{\text{lat.}}$$

Is defined by

$$f_a^* \psi_1(z) := \int_{x \in T^n} \rho_a(z-x) U(x,z)^{-1} \psi_1(x) d^n x.$$

Note) $f_a^* f_a$ is not the identity but smeared around nearest-neighbor sites.
(The gauge invariance is maintained by the Wilson lines.)

# Continuum limit of $f_a^*$ $f_a$

1. For arbitrary $\phi^{\mathrm{lat.}}$

   $\lim_{a\to 0} f_a \phi^{\mathrm{lat.}}$ weakly converges to a $^\exists \phi_0^{\mathrm{cont.}} \in L_1^2$

   where $L_1^2$ is a subspace of $H^{\mathrm{cont.}}$ where the elements and their first derivatives are square integrable.

2. $\lim_{a\to 0} f_a \gamma(D_W + m)\phi^{\mathrm{lat.}}$ weakly converges to $\gamma(D+m)\phi_0^{\mathrm{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a \phi^{\mathrm{lat.}} - \phi^{\mathrm{lat.}}||_{L^2}^2 < ca^2 ||\phi^{\mathrm{lat.}}||_{L_1^2}^2$

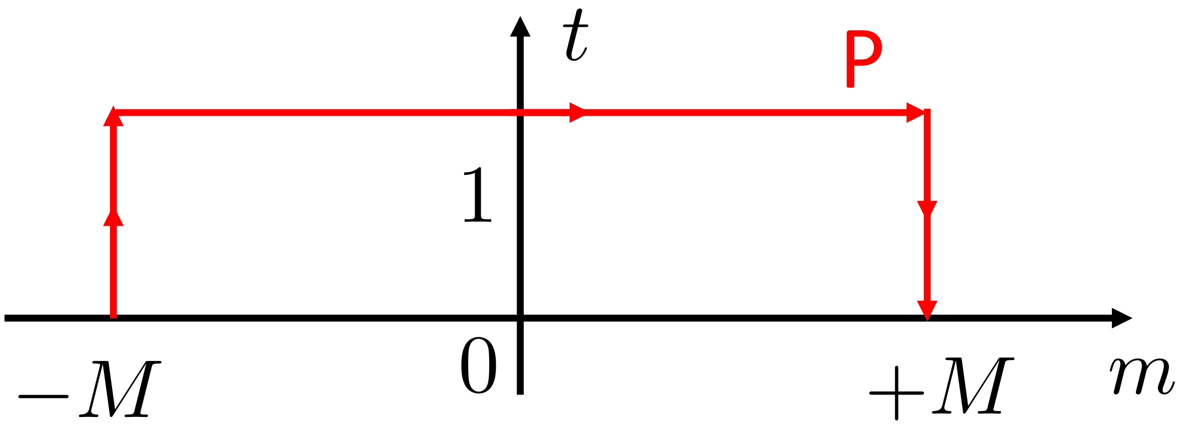
4. For any $\phi^{\mathrm{cont.}} \in L_1^2$, $\lim_{a\to 0} f_a f_a^* \phi_0^{\mathrm{cont.}} = \phi_0^{\mathrm{cont.}}$

# Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



# Main theorem

There exists a finite lattice spacing $a_0$ such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\rm cont.} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P
[which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D_{\rm cont.} + m), \ \ \gamma(D_W + m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D - M))^{\rm PV\ reg.} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

In our work, the proof is given by contradiction.

# Numerical test

We consider a two-dimensional square lattice (cont. limit= torus)
We set link variables as

$$U_y(x,y) = \exp\left[i\frac{2\pi Q(x-x_0)a}{L_1^2}\right]$$

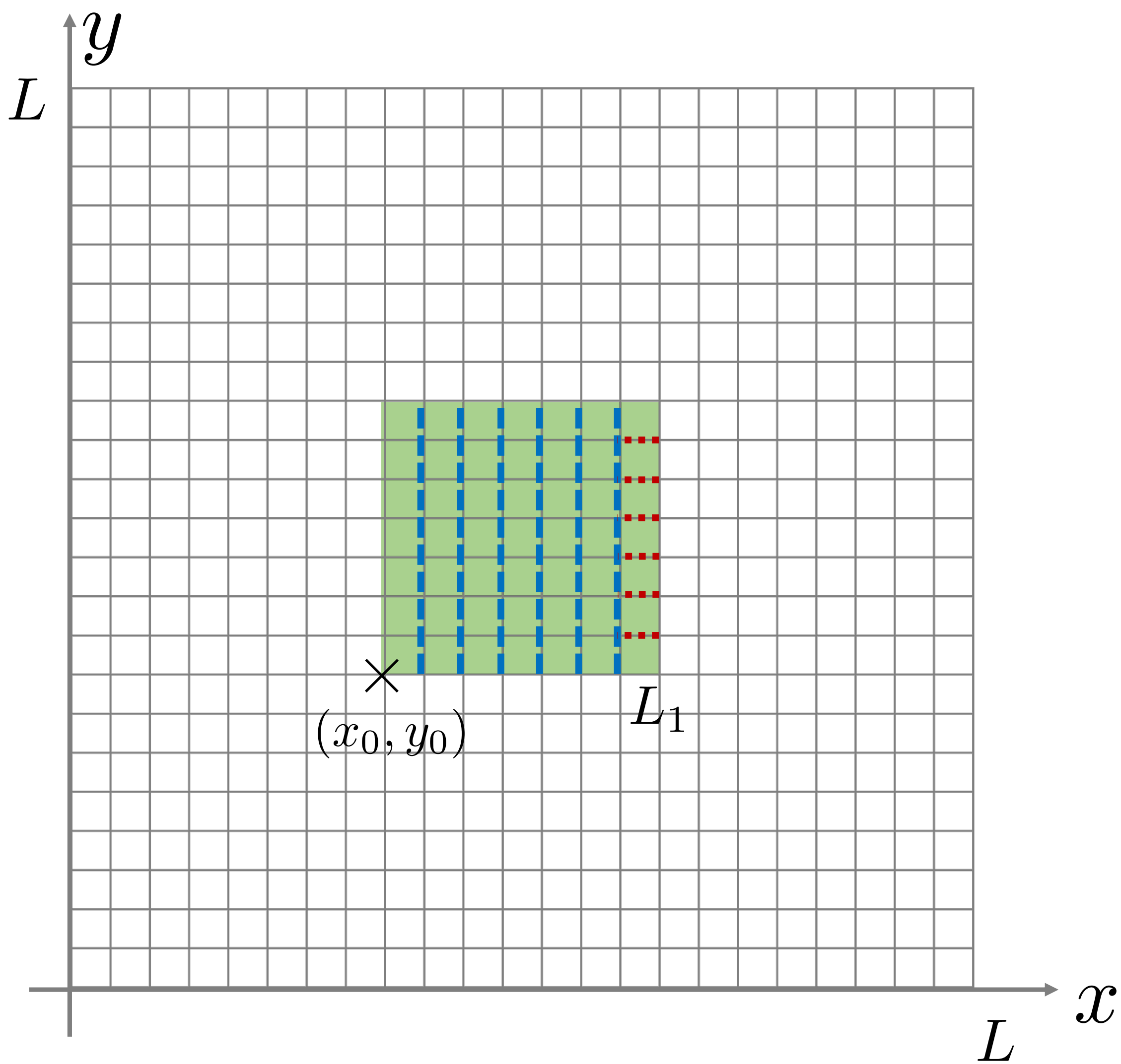
$$U_x(x,y) = \exp\left[-i\frac{2\pi Q(y-y_0)}{L_1}\right]$$

$$\text{others} = 1.$$

Then every green plaquette has a constant curvature

$$U_P(x,y) = \exp\left[i\frac{2\pi Q a^2}{L_1^2}\right]$$

so that geometrical index will be Q.




This constant curvature background can be extended to any even dimensions with SU(N) gauge connections [Cf. Hamanaka-Kajiura 2002].

# Massive Wilson Dirac

$$\gamma D_W(m) = \gamma\left[\sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f \nabla_i^b\right] + m\right]$$

$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x}) \quad a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

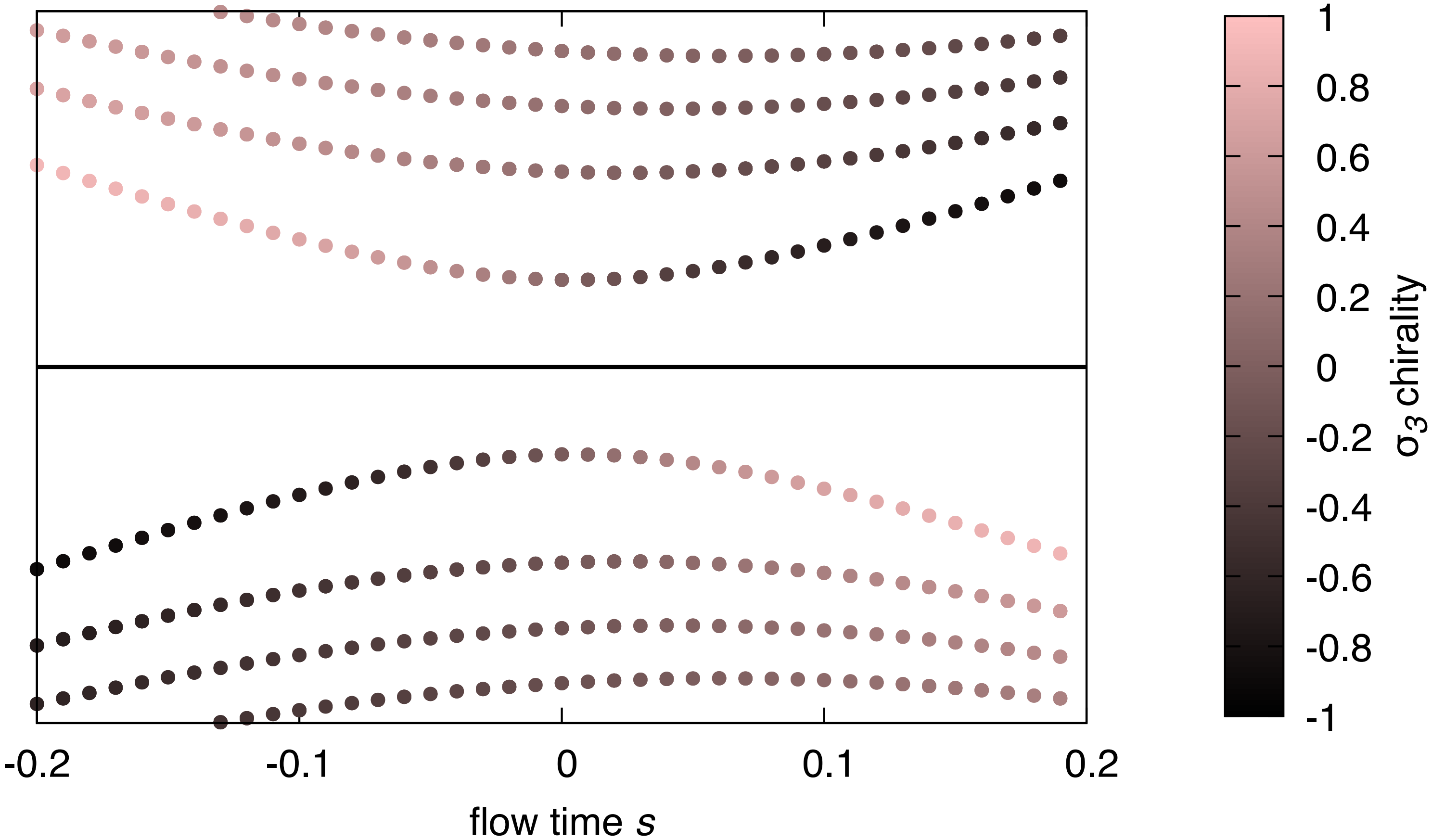
with periodic b.c. in x-direction and anti-periodic b.c. in y direction. We set L=33 and L1=10.

We compute near-zero eigen-spectrum changing the mass $-sM, \ -1 \le s \le +1$

# Wilson Dirac spectrum at Q=0

$$H_W(s) = \gamma(D_W - sM)$$

$$M = 1/a$$

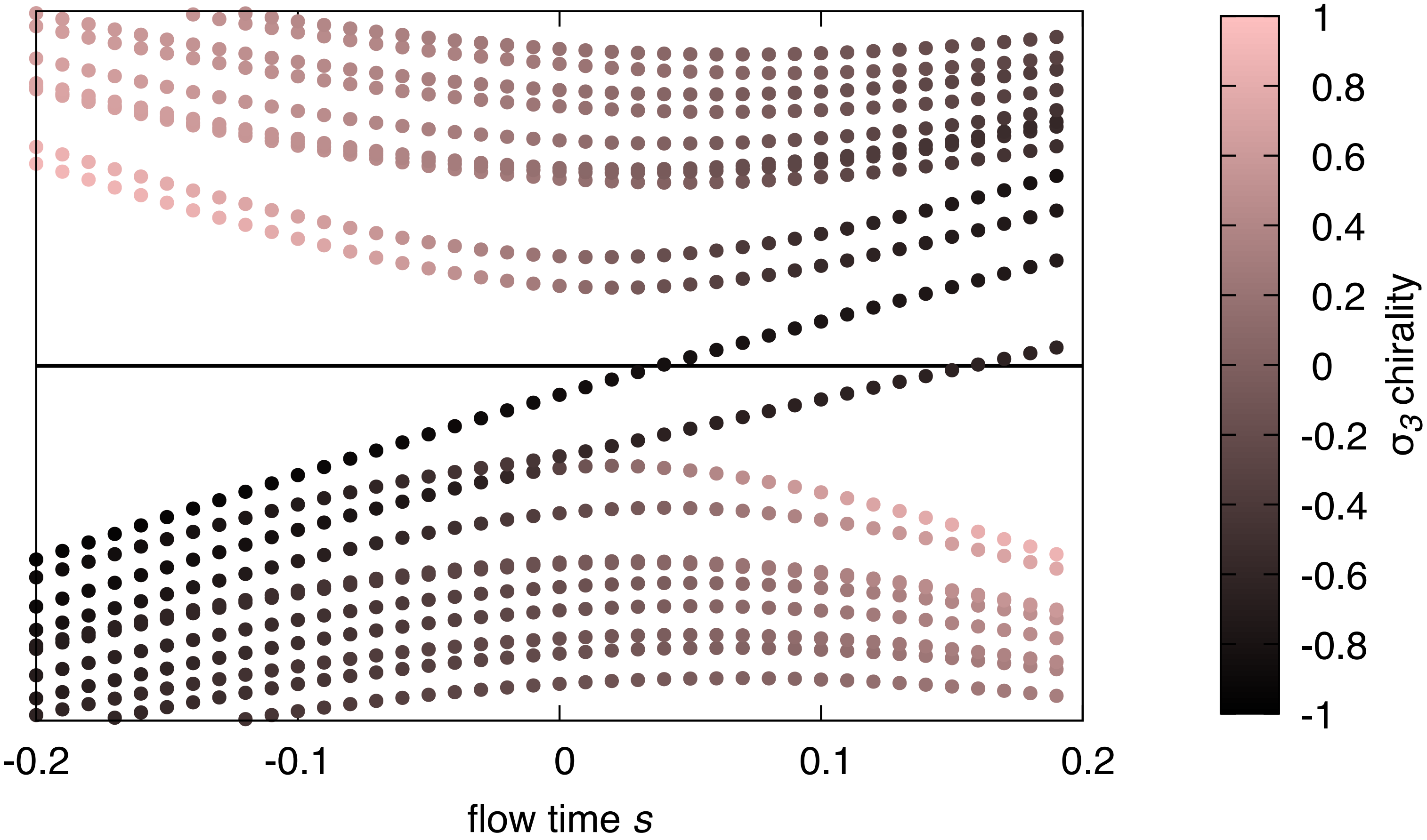



There is no zero crossing : index=0.

# Wilson Dirac spectrum at Q=-2

$$H_W(s) = \gamma(D_W - sM) \quad M = 1/a$$

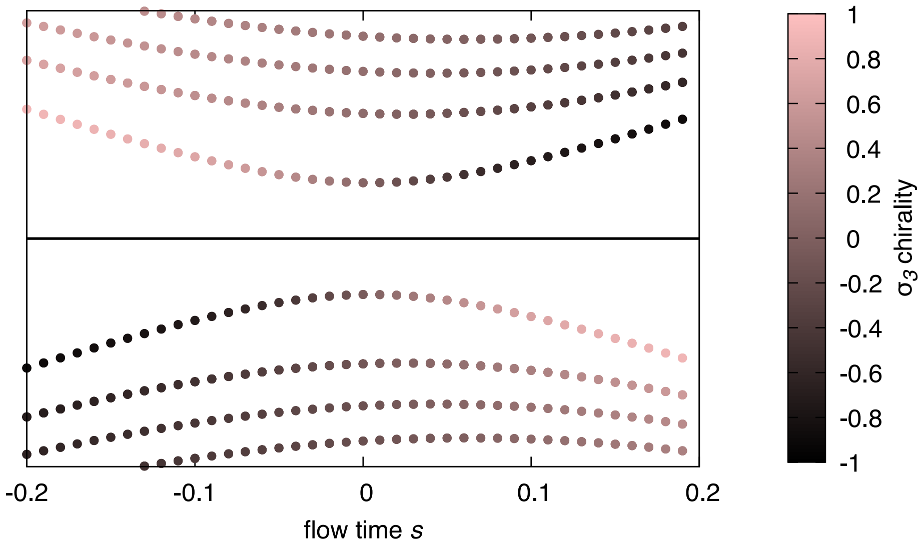
$$-\frac{1}{2}\eta(\gamma(D_W - M)) = -2$$



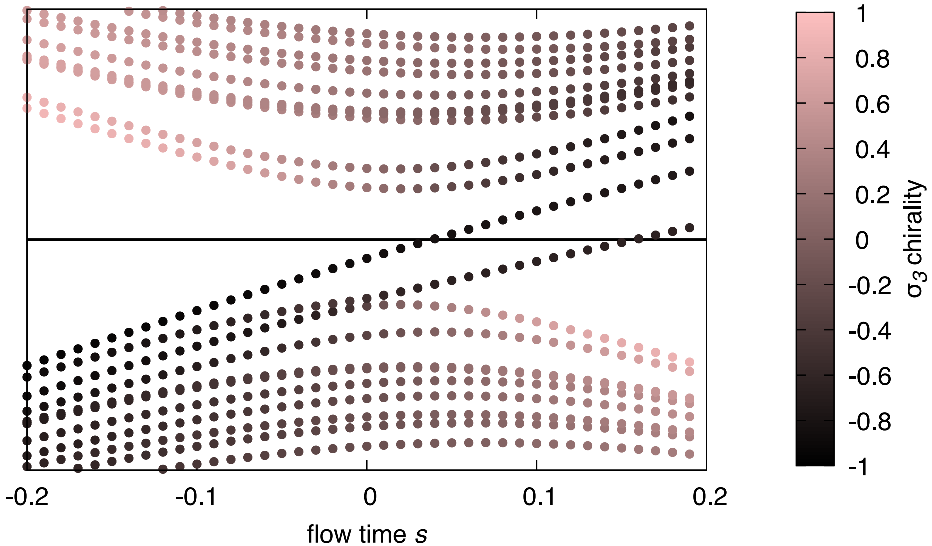

There is two crossings from negative to positive: index=-2.

Gradation represents the chirality expectation value: $\langle\lambda|\gamma|\lambda\rangle$

# Our lattice reproduces the Atiyah-Singer index theorem on a torus.




Index=Q=0




Index=Q=-2

This agrees with the overlap Dirac index.

# Contents

✓ 1. Introduction
   We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.
✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)
   great but equivalent to the eta invariant of the massive Wilson Dirac op.
✓ 3. K-theory
   classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.
✓ 4. Massless Dirac ($K^0$ group) vs. massive Dirac ($K^1$ group) in continuum
   Counting lines (massive, $K^1$) is easier than counting points (massless, $K^0$).
✓ 5. Main theorem on a lattice
   The proof is given by lattice-continuum combined Dirac operator, which is gapped.
6. Applications to a manifold with boundaries and the mod two version

7. Summary and discussion

Wilson Dirac operator is equally good as $D_{ov}$ to describe the index (or maybe better).

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

(so far) limited to even-dimensional flat torus.

By $K^1(I, \partial I)$ for sufficiently small lattice spacings

Suspension isomorphism

K theory knows how to extend the formulation to the systems (where chiral symmetry is absent or difficult ) with (curved) boundaries and/or mod-two version in arbitrary dimensions.

# Atiyah-Patodi-Singer index on a manifold with boundaries

Periodic b.c.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

Open b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D_{DW}^{\mathrm{cont.}})) = \mathrm{Ind}_{\mathrm{APS}} D^{\mathrm{cont.}}$$

[perturbative test by F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019 Mathematical proof ongoing.].

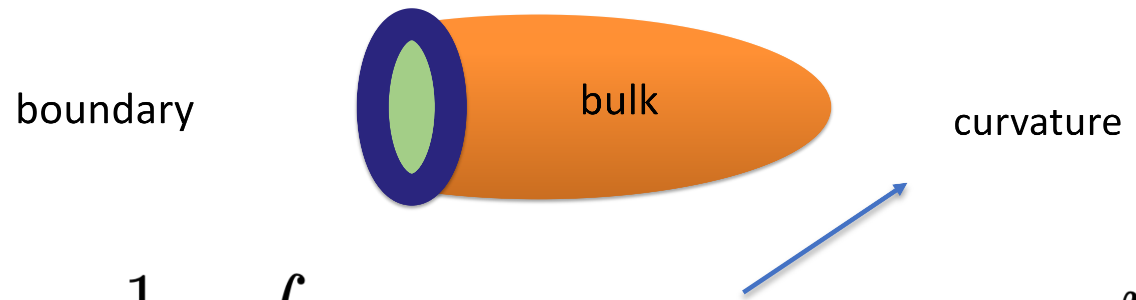
Atiyah-Patodi-Singer(APS) index !

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

Kaplan's DWF gives the same index.

Cf.) overlap Dirac op. is missing because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

# Atiyah-Patodi-Singer index theorem [1975]




$$Ind(D_{\rm APS}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

* example of 4-dimensional
flat Euclidean space with boundary at $x_4$=0.

# Numerical test on a 2D disk

[Cf. Aoki-F, 2022,2023]

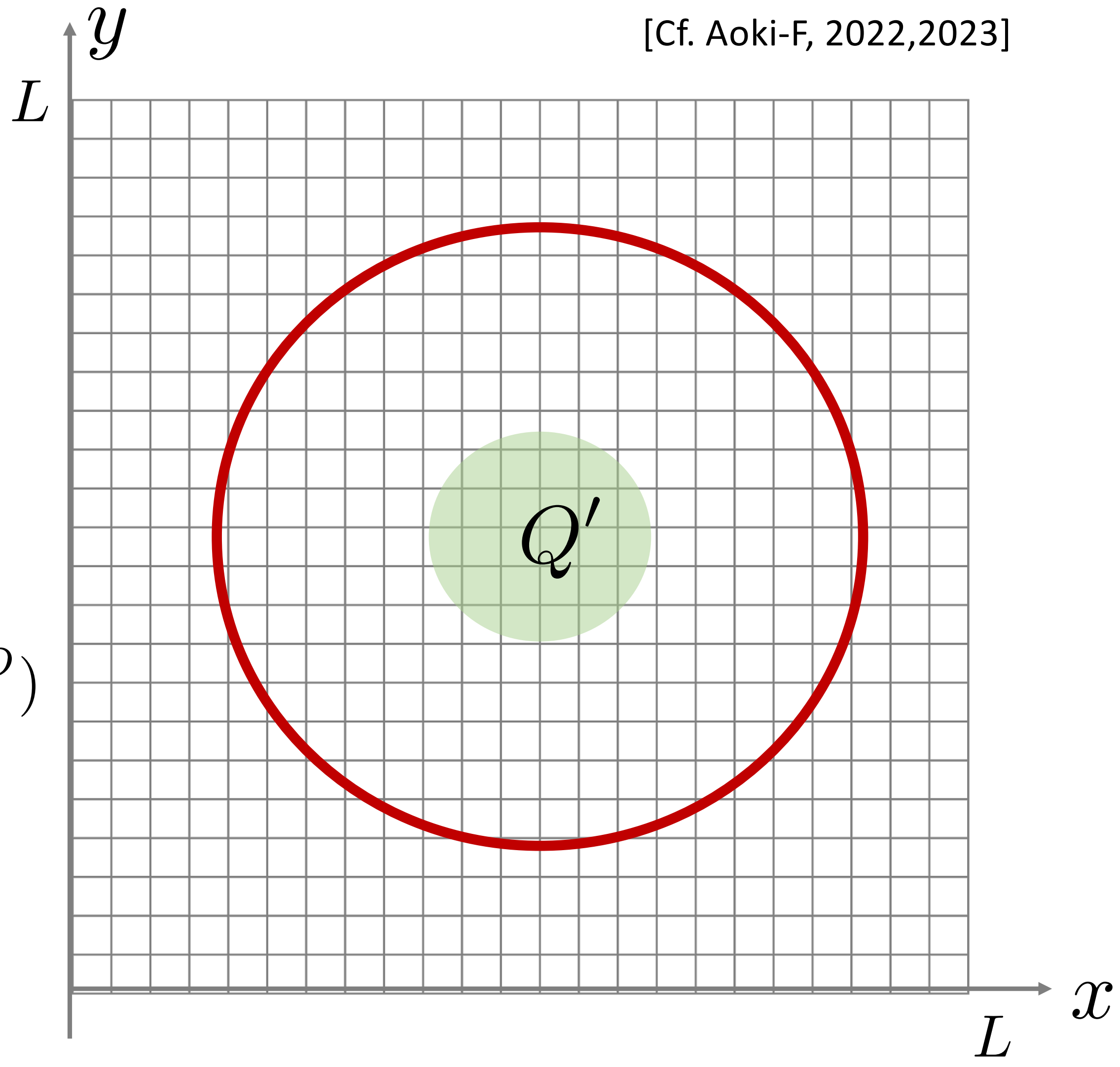
We put a circular curved domain-wall : m=-s/a inside, m=+1/a outside and change s from -1 to 1.

We put U(1) flux Q' and numerically check if the APS index theorem

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

holds or not.

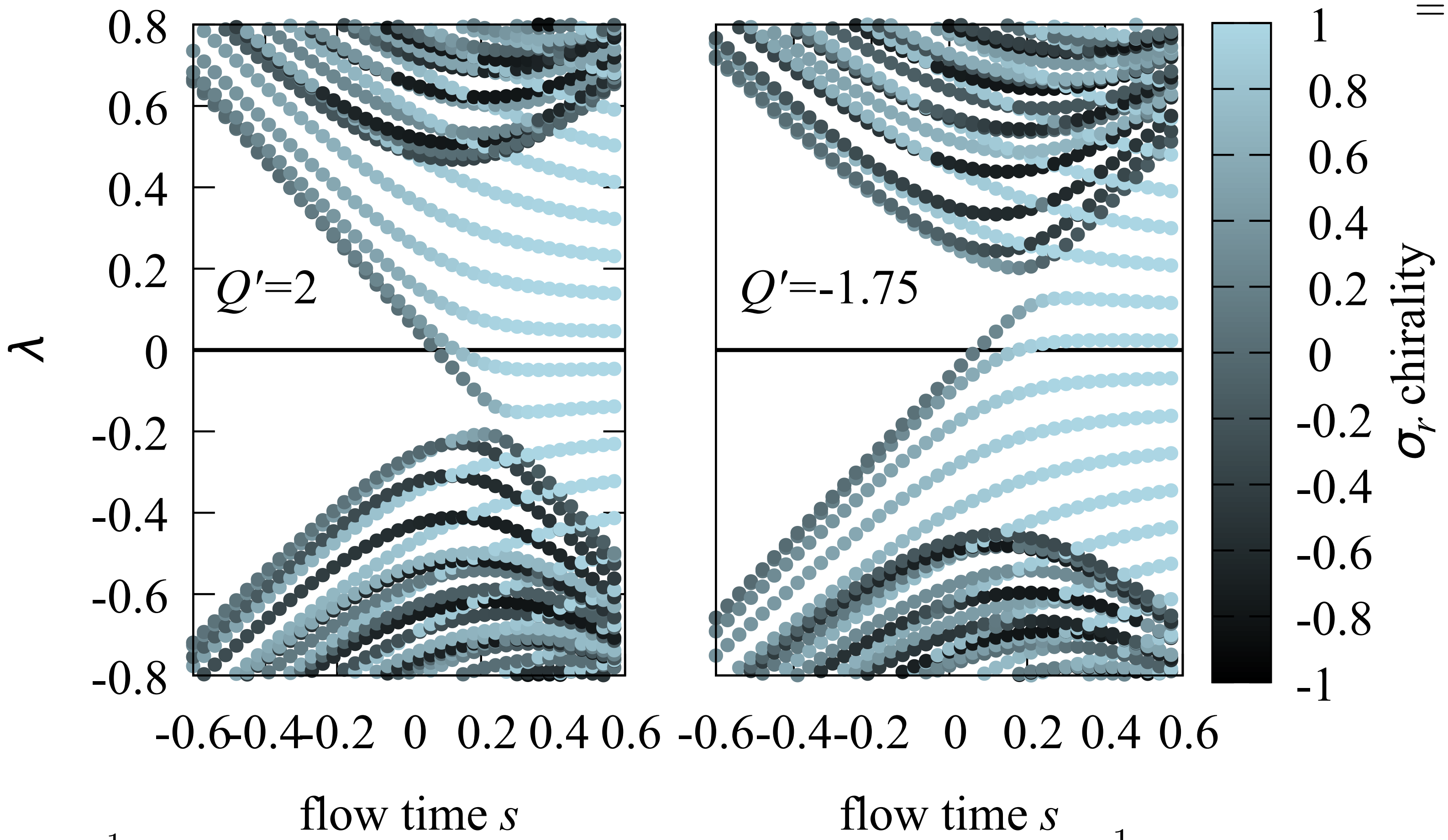
L=33, DW radius=10, flux radius=6.



# Dirac spectrum on a 2D disk

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$




$-\frac{1}{2}\eta(iD^{1D}) = 0.$

$-\frac{1}{2}\eta(iD^{1D}) = -0.25$

Edge-localized chiral : $\sigma_r = (\sigma_1 x + \sigma_2 y)/r \sim 1$ modes appear on the 1-dimensional circle domain-wall = the source of boundary eta invariant.

Consistent with the APS Index theorem.

# The boundary eta invariant (details)

[ Aoki-F, 2022,2023]

$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

Continuum result for
1D Dirac eigenvalues on a circle

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

$$\lambda_j = \frac{1}{r_0}\left(j + \frac{1}{2} - Q'\right)$$

Gravity
(Spin$^c$ connection)

Aharonov-Bohm effect

$$-\frac{1}{2}\eta(iD^{1\mathrm{D}}) = -\frac{1}{2}\lim_{s\to 0}\sum_{\lambda}\frac{\lambda_j}{|\lambda_j|^{1+s}} = [Q'] - Q'$$

$$\mathrm{Ind}D_{\mathrm{APS}} = [Q']$$ Gauss symbol: the biggest integer $\leq Q'$

$$[2] = 2, \quad [-1.75] = -2.$$

# Real Dirac operators and the mod-two index

For complex Dirac operators, we have shown

$$K^1(I,\partial I) \Rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D-M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we obtain the mod-2 spectral flow:

$$KO^0(I,\partial I) \Rightarrow -\frac{1}{2}\left[1-\mathrm{sgn}\det\left(\frac{D_W-M}{D_W+M}\right)\right] = -\frac{1}{2}\left[1-\mathrm{sgn}\det\left(\frac{D_{\mathrm{cont.}}-M}{D_{\mathrm{cont.}}+M}\right)\right]$$

$$= \mathrm{Ind}_{\mathrm{mod-two}} D_{\mathrm{cont.}}$$

[F, Furuta, Matsuki, Matuso, Onogi, Yamaguchi, Yamashita 2020].

But there is no overlap Dirac counterpart.

# Mod-two index and mod-two spectral flow

Two types of the mod-two index

1. number of zero modes of real anti-Hermitian operator

$$D \in KO^{-1}(\text{point})$$

2. number of zero mode pairs of real anti-Hermitian operator

$$\tau_1 \otimes D \in KO^{-2}(\text{point})$$

For both cases, we can consider massive operator family

$$D_s = \tau_1 \otimes D - i\tau_2 \otimes sM \in KO^0(I, \partial I)$$

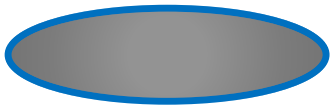
and the mod-two spectral flow = number of pairs of zero-crossings agrees with the original index.

# Numerial test for Majorana $S^1$ domain-wall fermion

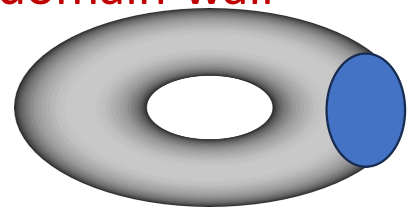
Free Wilson Dirac operator is real:

$$iH_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 (W + M(x))$$

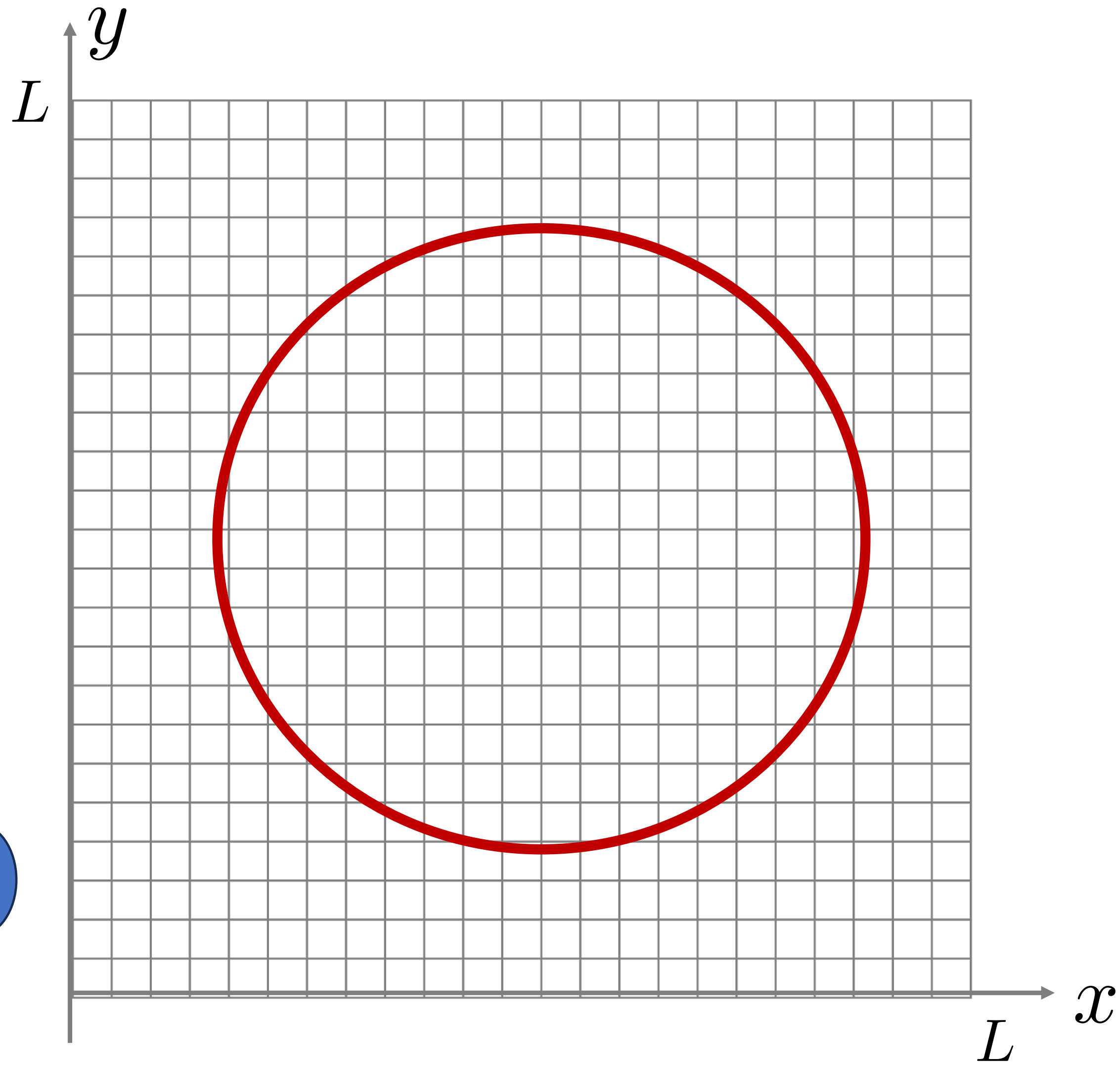
Mass change inside the domain-wall
=disk



Mass change outside the domain-wall
= torus with a $S^1$ hole.

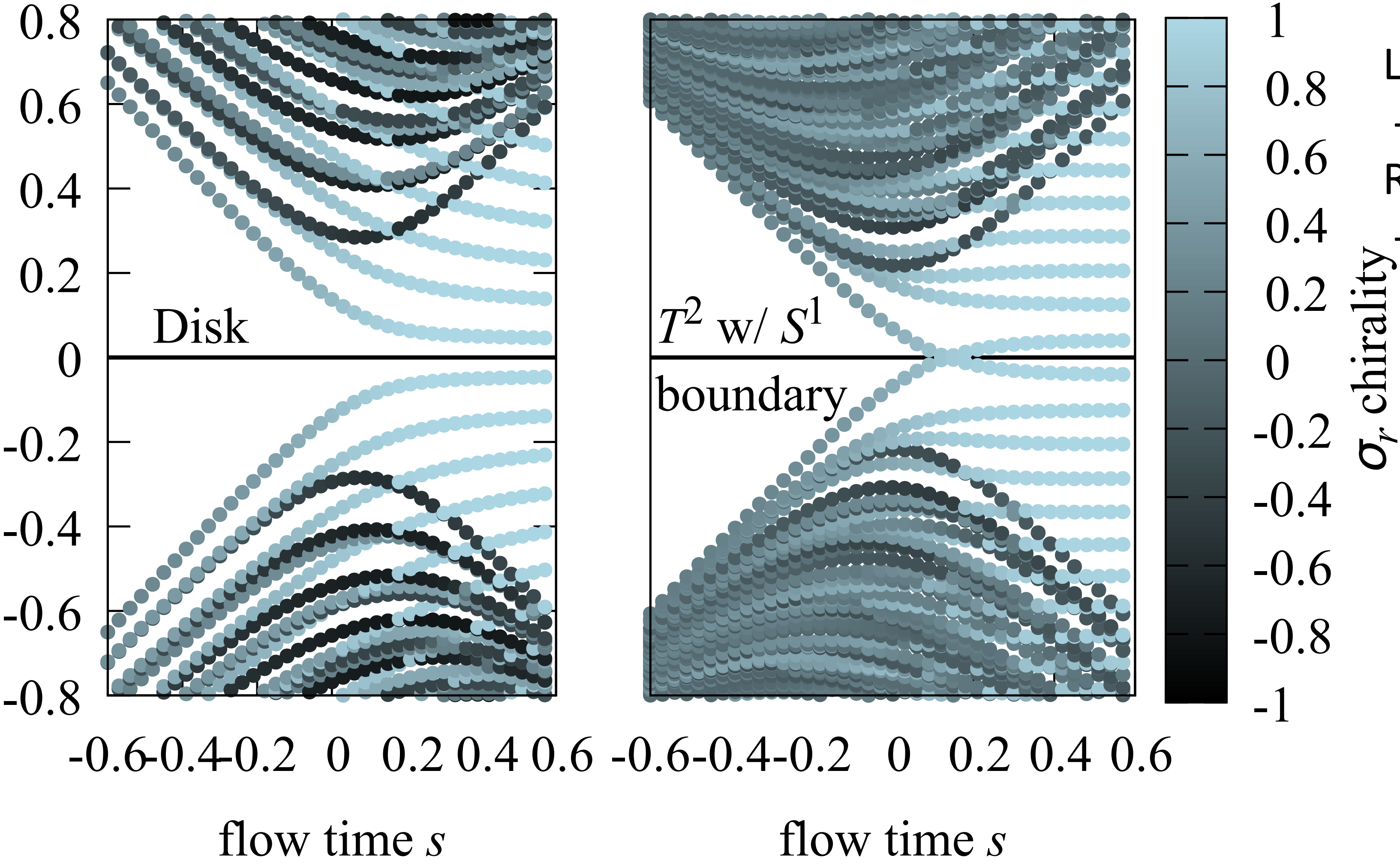


The continuum mod-two APS index = 0
and 1 respectively.



# Majorana Dirac spectrum

$$iH_m = \sigma_1\partial_x + \sigma_3\partial_y + i\sigma_2 m(s,r)$$




Left panel :
-sM inside $S^1$ DW
Right :
-sM outside $S^1$ DW

mod-two spectral flow agrees with the mod-two APS index.

# Contents

# Summary

The massive Wilson Dirac operator can be identified as a mathematical object in K-theory and the associated spectral flows describe various index formulas.

In our formulation,

- chiral symmetry (GW relation) is NOT necessary,
  (besides, it agrees with the overlap index on periodic lattices)
- boundaries can be introduced by domain-walls,
- domain-walls can be flat/curved (with gravitational background),
- formulated in arbitrary dimensions,
- standard/mod-two versions treated in a unified way.

# Outlook

* “existence” of sufficiently small lattice spacing -> more clear-cut admissibility condition?

* flat bulk + curved domain-wall -> curved bulk and domain-wall by higher codimensional defects?

* Unorientable manifolds? (Araki, F, Onogi, Yamaguchi ongoing)

* How about physicist friendly eta invariant ?

# Backup slides

# Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,

a constant c exists such that

$$||D_i \phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large, D is also large.

This property is nontrivial on a lattice.

$$||\nabla_i^f \phi||^2 \leq c(||\phi||^2 + ||D_W \phi||^2)$$

Without Wilson term, doubler modes would have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important to make the Dirac operator elliptic.

# Proof (by contradiction)

Assume
$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

# Continuum limit

Multiplying $\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\mathrm{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$$\hat{D}^2_\infty = D^2_{\mathrm{cont.}} + m^2_\infty + t^2_\infty$$

requires

$$m_\infty = t_\infty = 0.$$

$u_\infty, \quad v_\infty$ are

$L^2_1$ weakly convergent

=

$L^2$ strongly convergent

(Rellich's theorem)

Contradiction with $m^2 + t^2 > 0$ along the path P.

# What are the weak convergence and strong convergence?

The sequence $v_j$ weakly converges to $v_\infty$

when for arbitrary $w$

$$\lim_{j\to\infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j\to\infty} (v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich's theorem:

$L^2_1$ weak convergence = $L^2$ convergence