Generalized global symmetry and application to Spontaneous symmetry breaking and QCD phase diagram

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Outline

Generalized global symmetries

- ordinary symmetry
- higher form symmetries and non-invertible symmetry

Application

- Spontaneous symmetry breaking
- QCD phase diagram

Summary

Ordinary symmetry in $(d+1)$ dimensions

Ex) $U(1)$ **symmetry**

$U(1)$ **charge:** $Q = \int d^d x j^0 = \int_{M^d} j$

Time independence: $\frac{d}{dt}Q = \int d^d x \partial_0 j^0 = -\int d^d x \nabla_i j^i = 0$

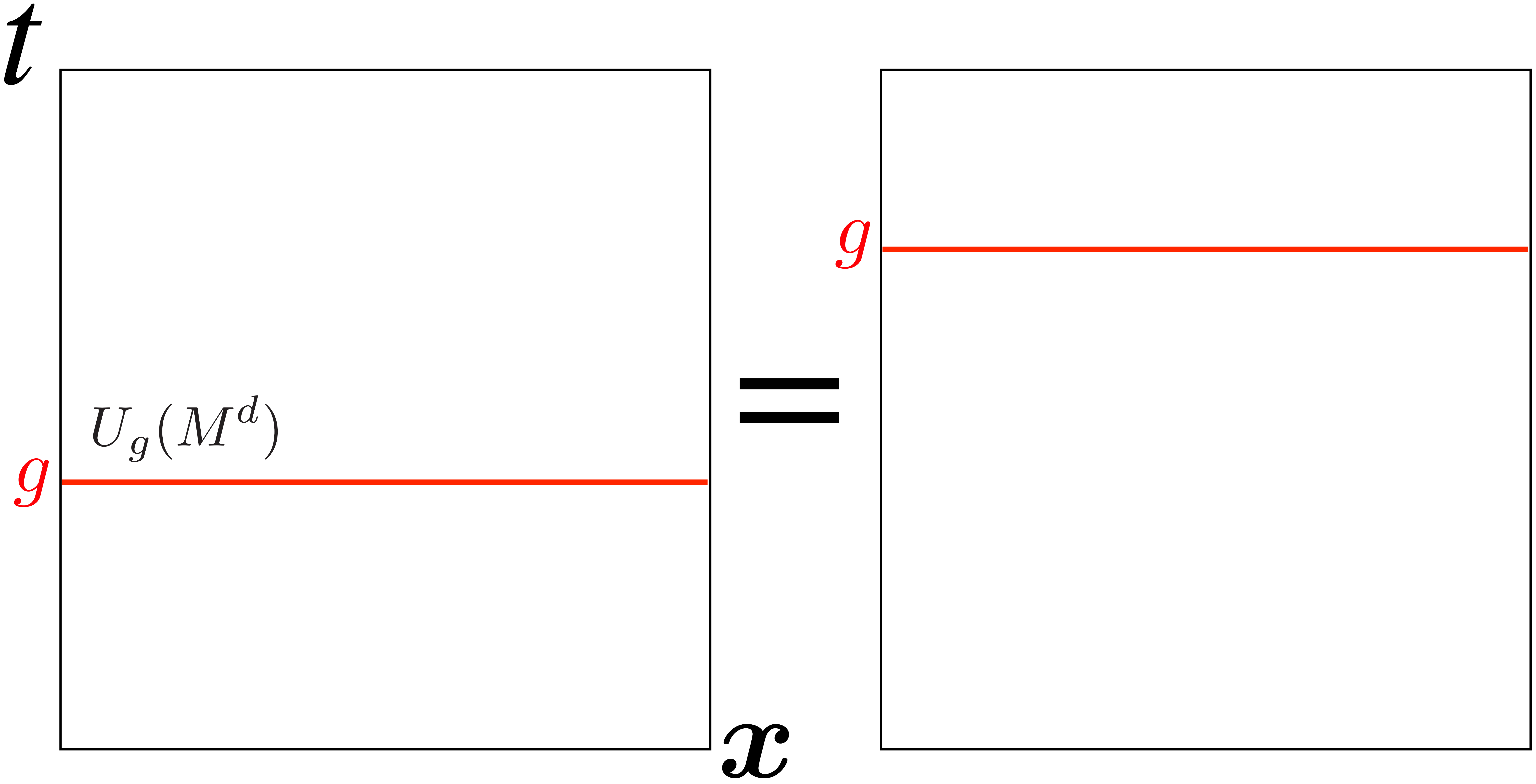
Unitary operator: $U_g(M^d) = e^{i\alpha Q} \quad (g = e^{i\alpha})$

Group law: $U_g(M^d) U'_g(M^d) = U_{gg'}(M^d)$

Charged object : $\varphi(x)$

Charged object : $U_g(M^d)\varphi(x)U_g^{-1}(M^d) = e^{-i\alpha}\varphi(x) = R(g)\varphi(x)$

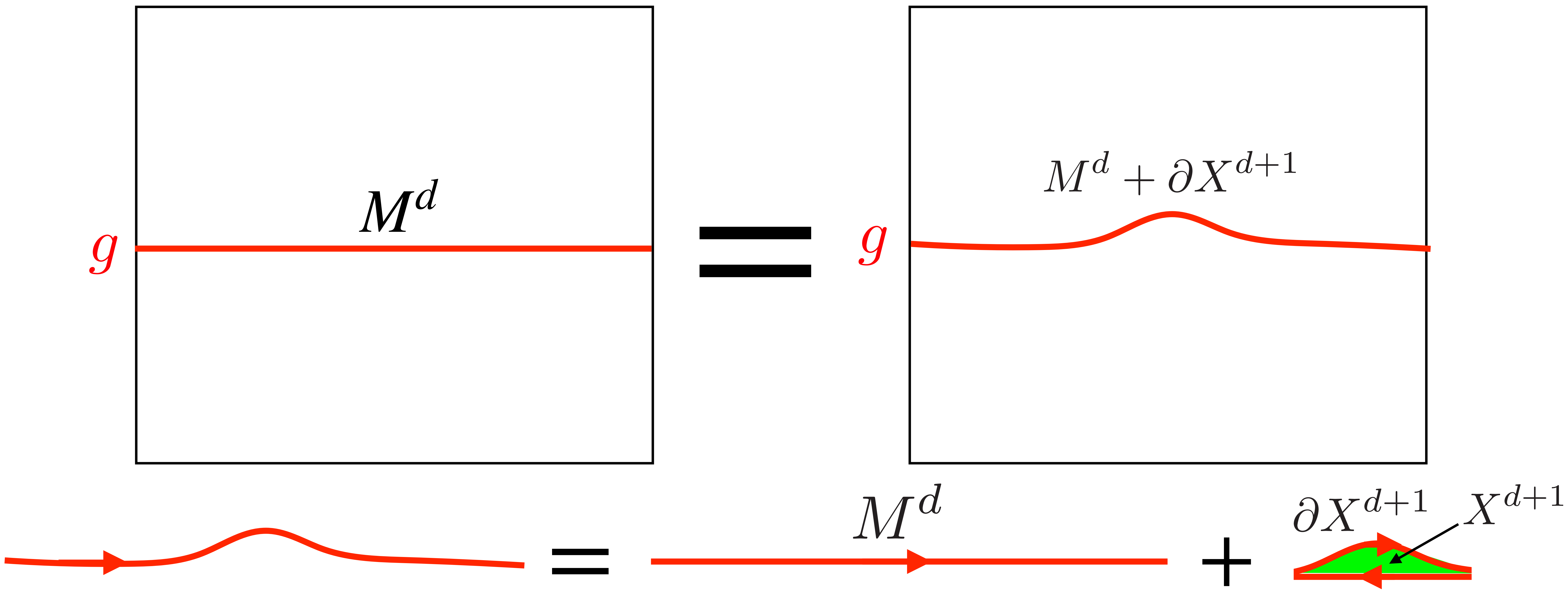
Graphical representation



Time independence

Graphical representation

Symmetry generator is topological



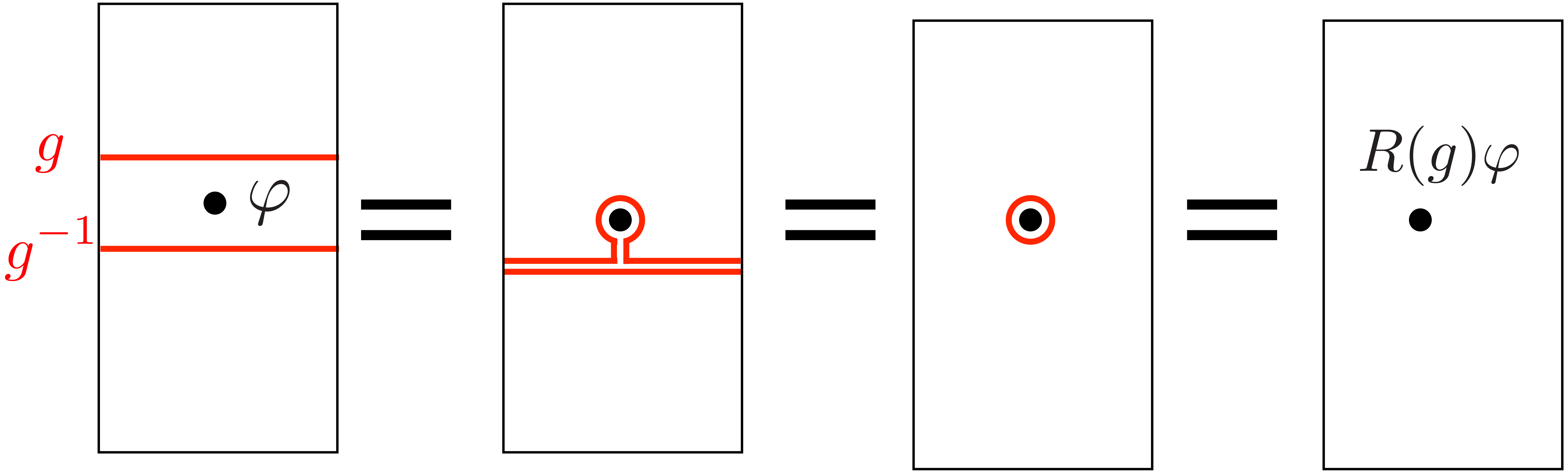
$$\int_{M^d+\partial X} j = \int_{M^d} j + \int_{\partial X} j = \int_{M^d} j + \int_X dj = \int_{M^d} j$$

Graphical representation

Charged object

$$U_g \varphi(x) U_{g^{-1}} = R(g)\varphi(x)$$

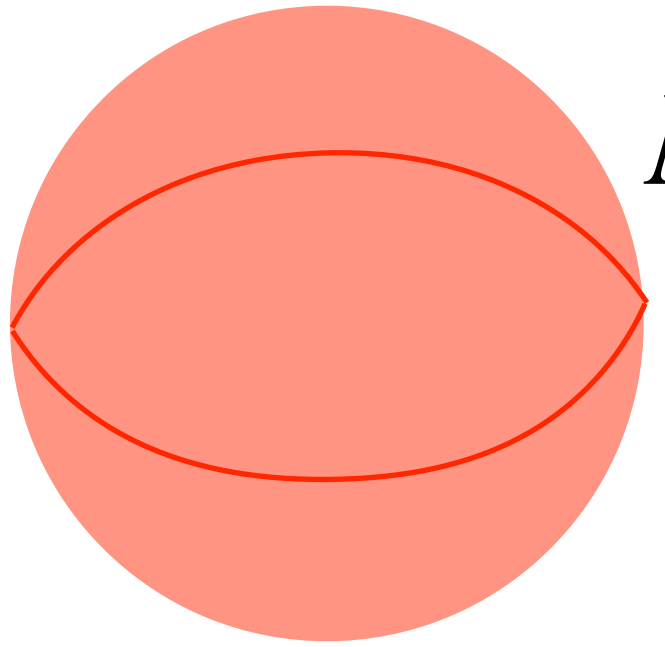
representation matrix



Brief summary

Symmetry generators

= d dimensional topological objects labeled by group elements

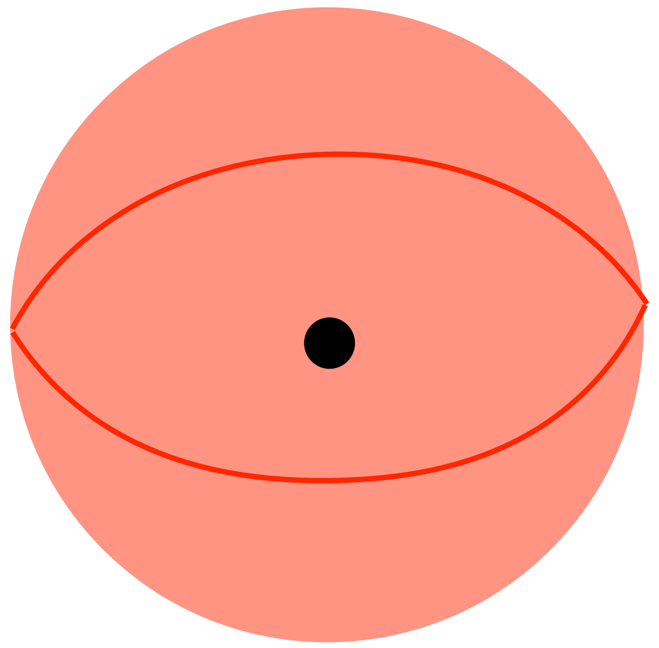
$U_g(M^d)$  M^d

Charged objects

= 0-dimensional objects labeled by representation of G

$\bullet\,\varphi_\rho(x)$

Charged object transforms under G

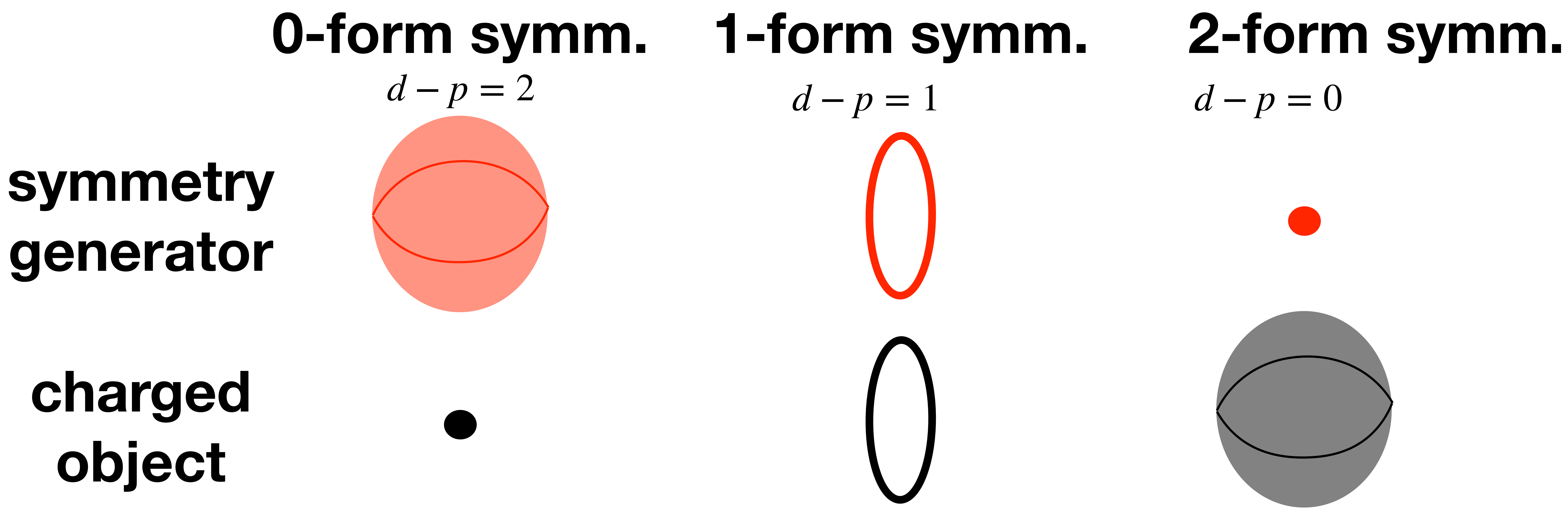
 $= R_\rho(g)\ \bullet$

p-form symmetry

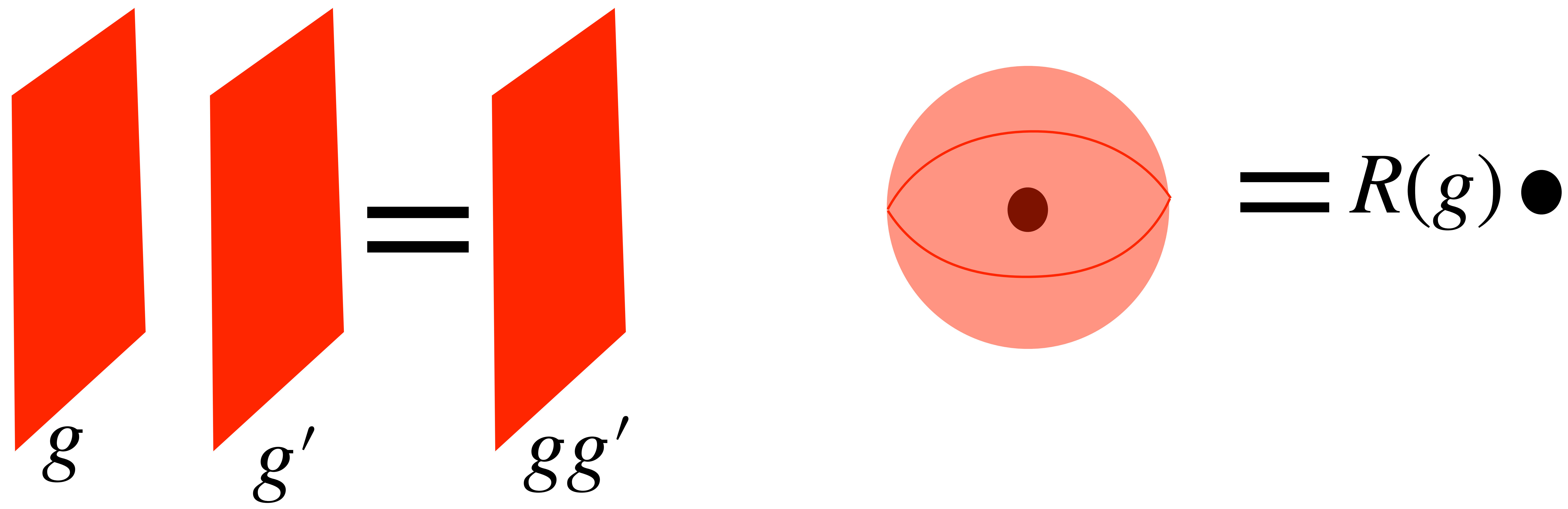
Charged object: p dimensional object

Symmetry generators:
$(d-p)$ dimensional topological objects labeled by group elements.

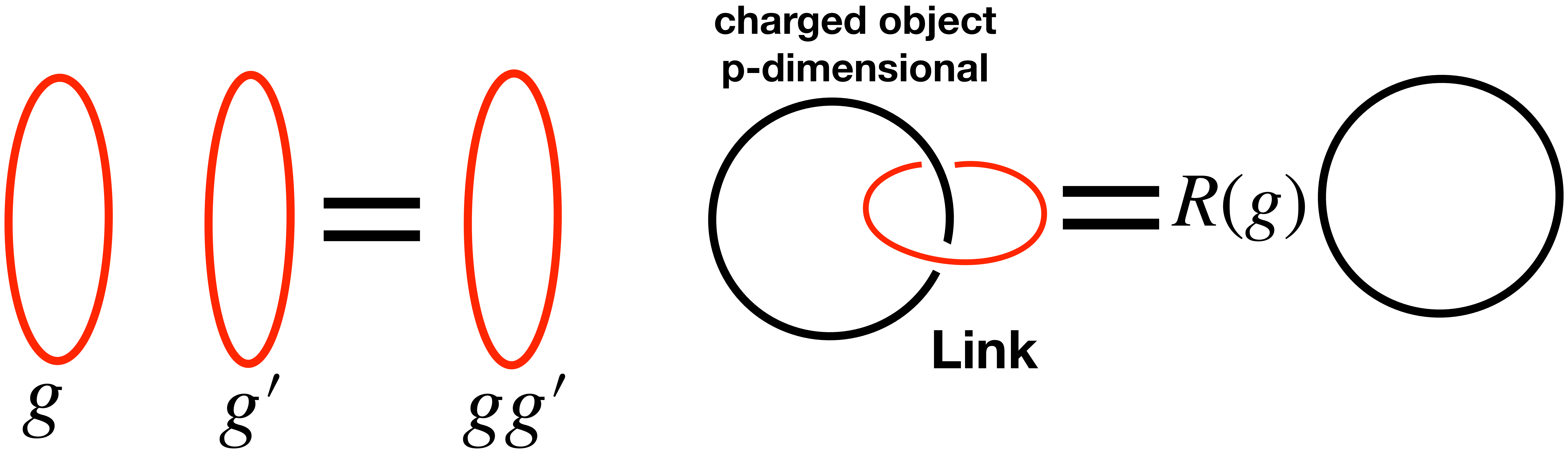
Ex) In 2+1 dimensions



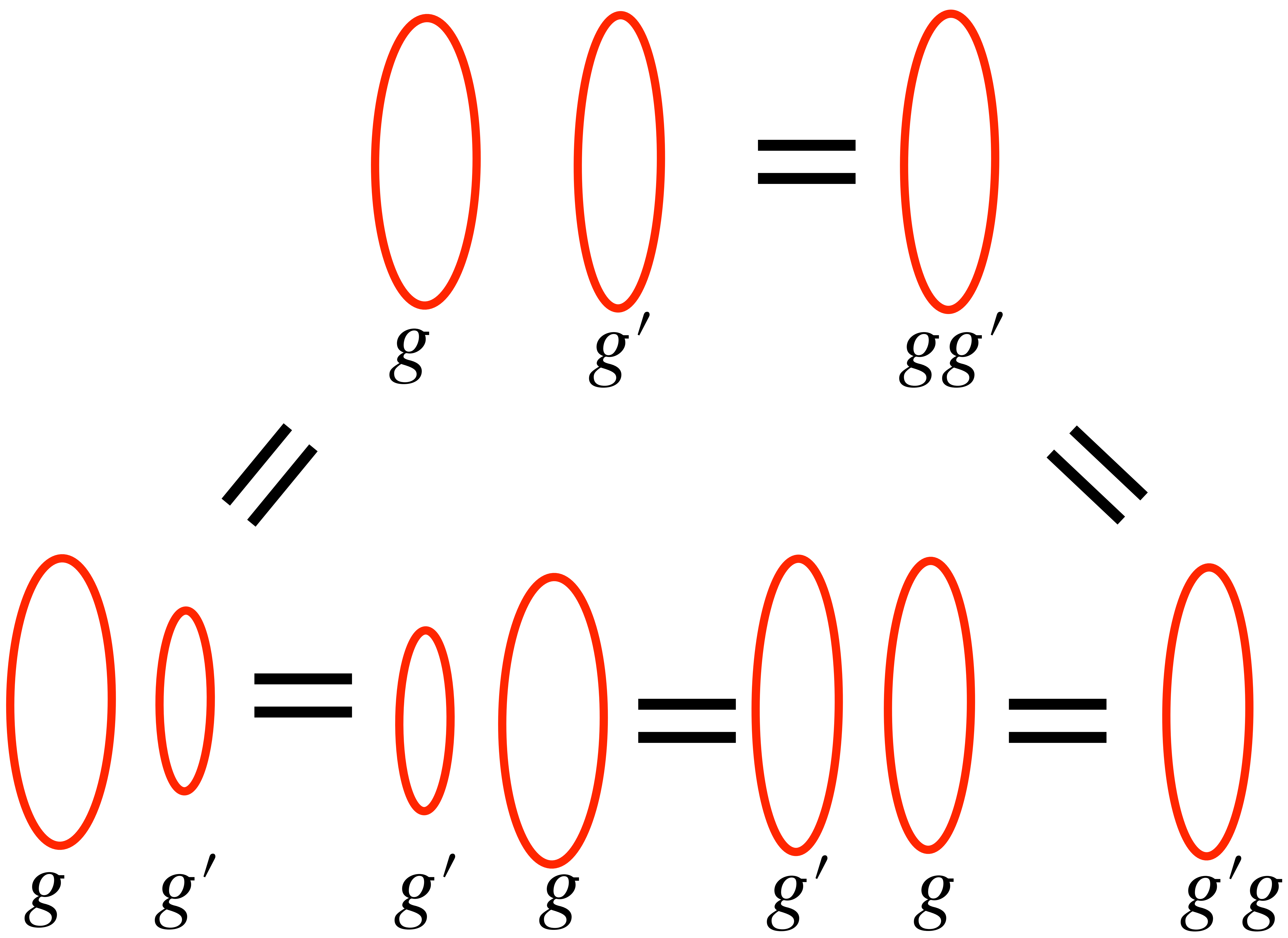
0-form symmetry



p-form symmetry



p-form symmetry($p \geq 1$) is abelian



Ex) U(1) gauge theory

$$S = -\int d^4x \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} = -\int \frac{1}{2e^2} f \wedge \star f$$ **where** $f = da$

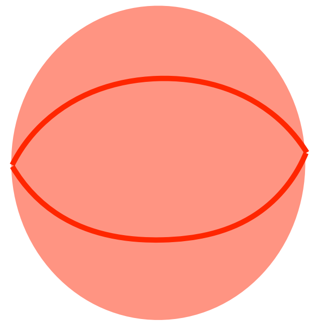
Maxwell equations

$$\partial_\mu f^{\mu\nu} = 0 \quad \Rightarrow \quad d \star f = 0$$

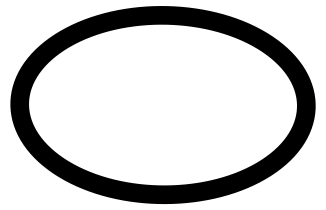
$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu f_{\nu\rho} = 0 \quad \Rightarrow \quad df = 0$$

Conservation of electric and magnetic fluxes

$U(1)^{[1]}_E \times U(1)^{[1]}_M$ **symmetries**

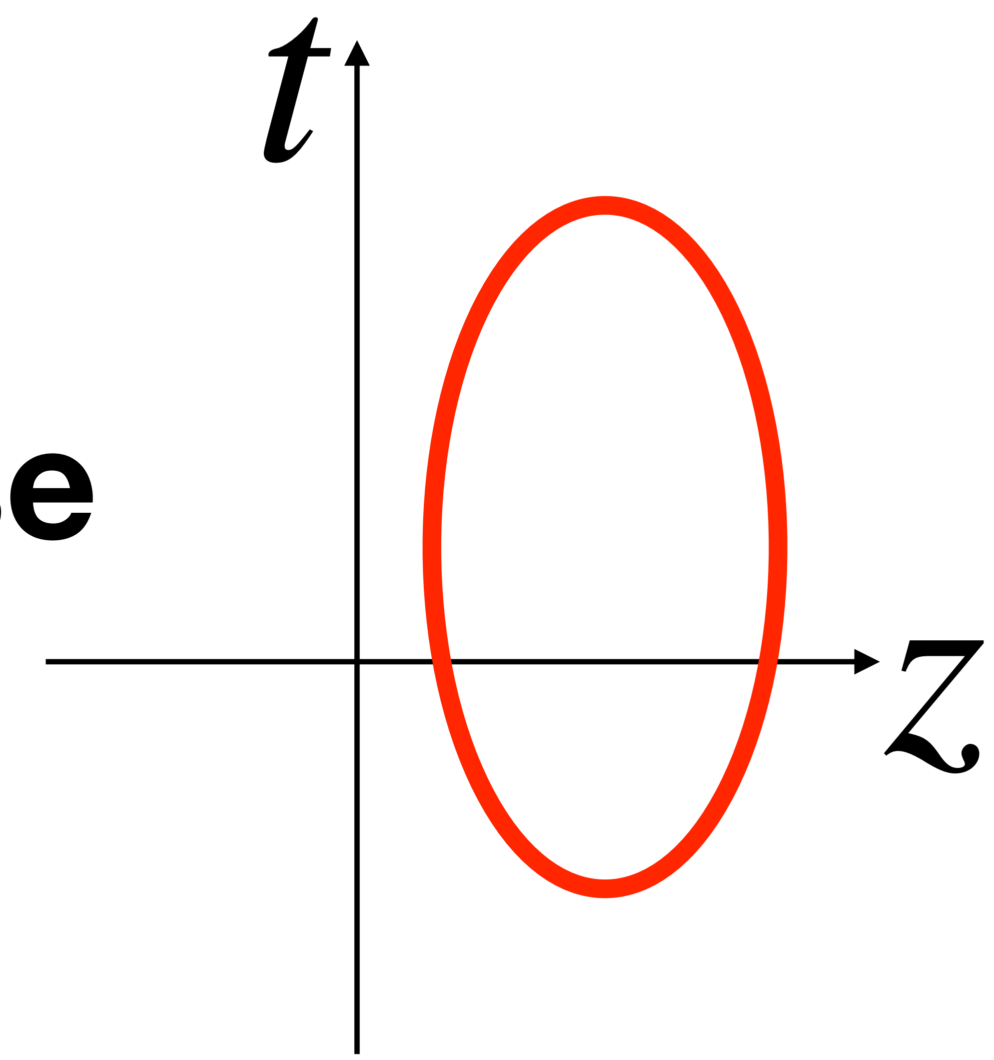


$$U_E = e^{i \frac{\theta_E}{e^2} \int_S \star f} \qquad U_M = e^{i \frac{\theta_M}{2\pi} \int_S f}$$

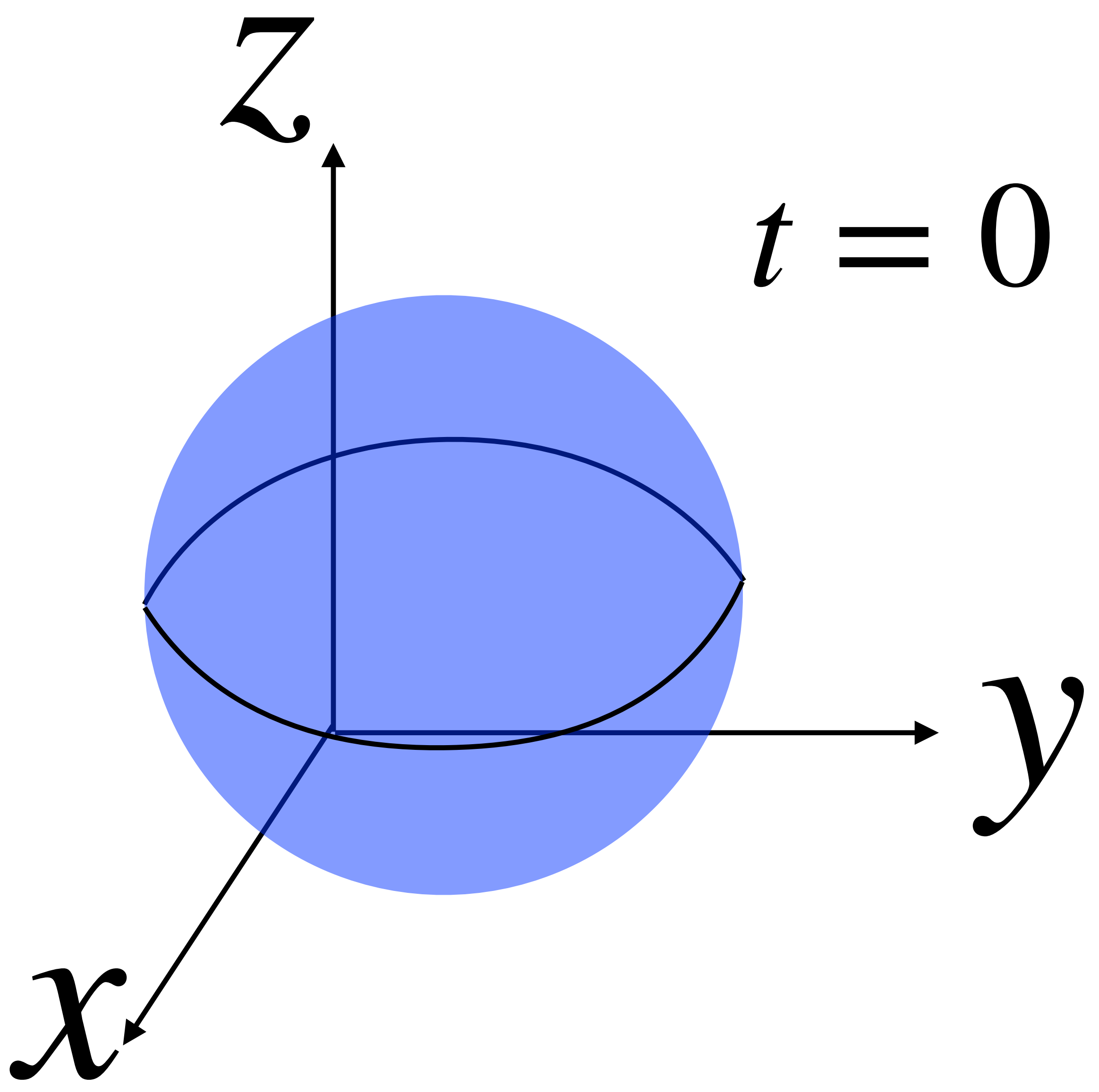


$$W = e^{i \int_C a} \qquad H = e^{i \int_C \tilde{a}}$$

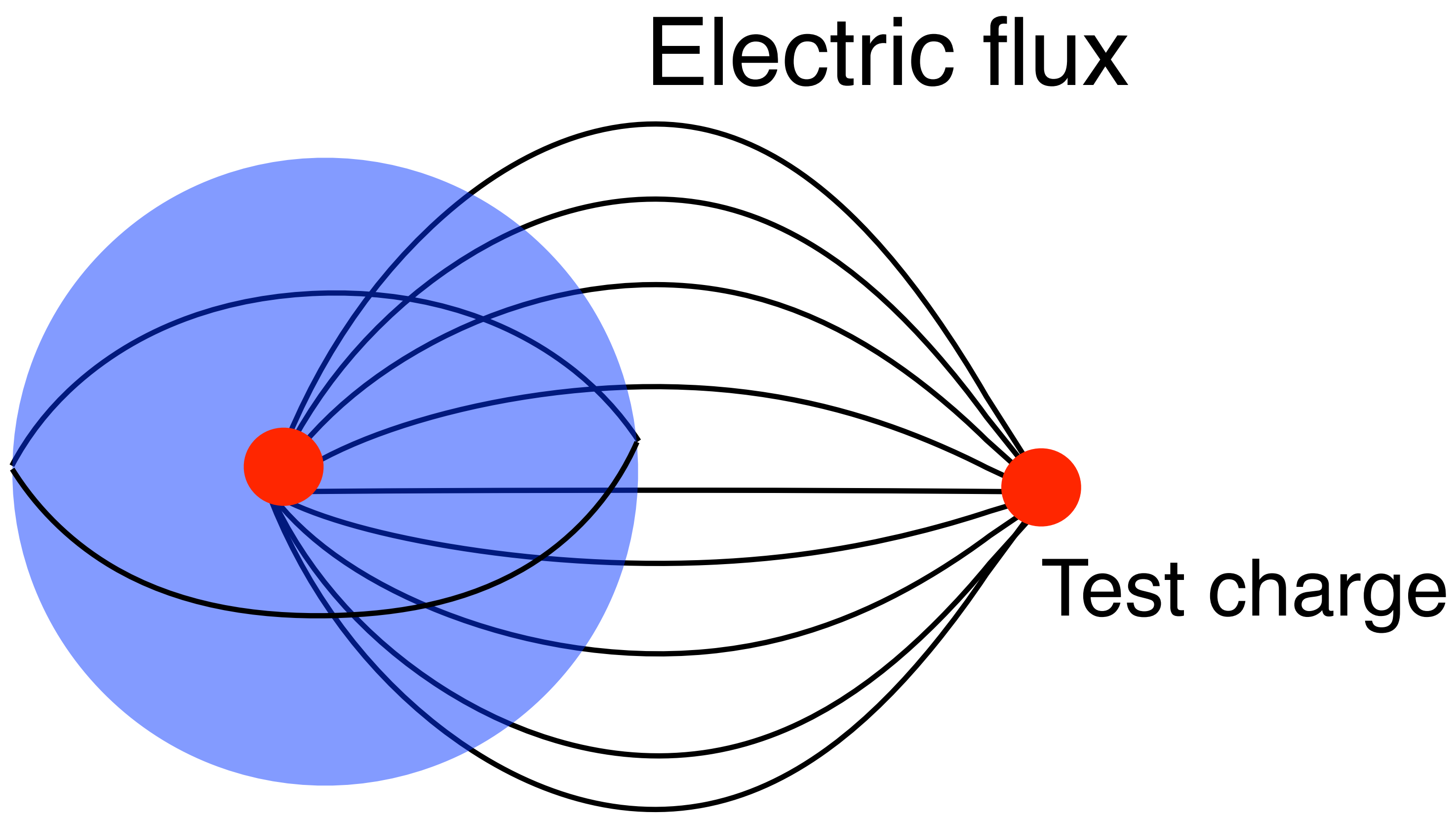
Let's choose



and

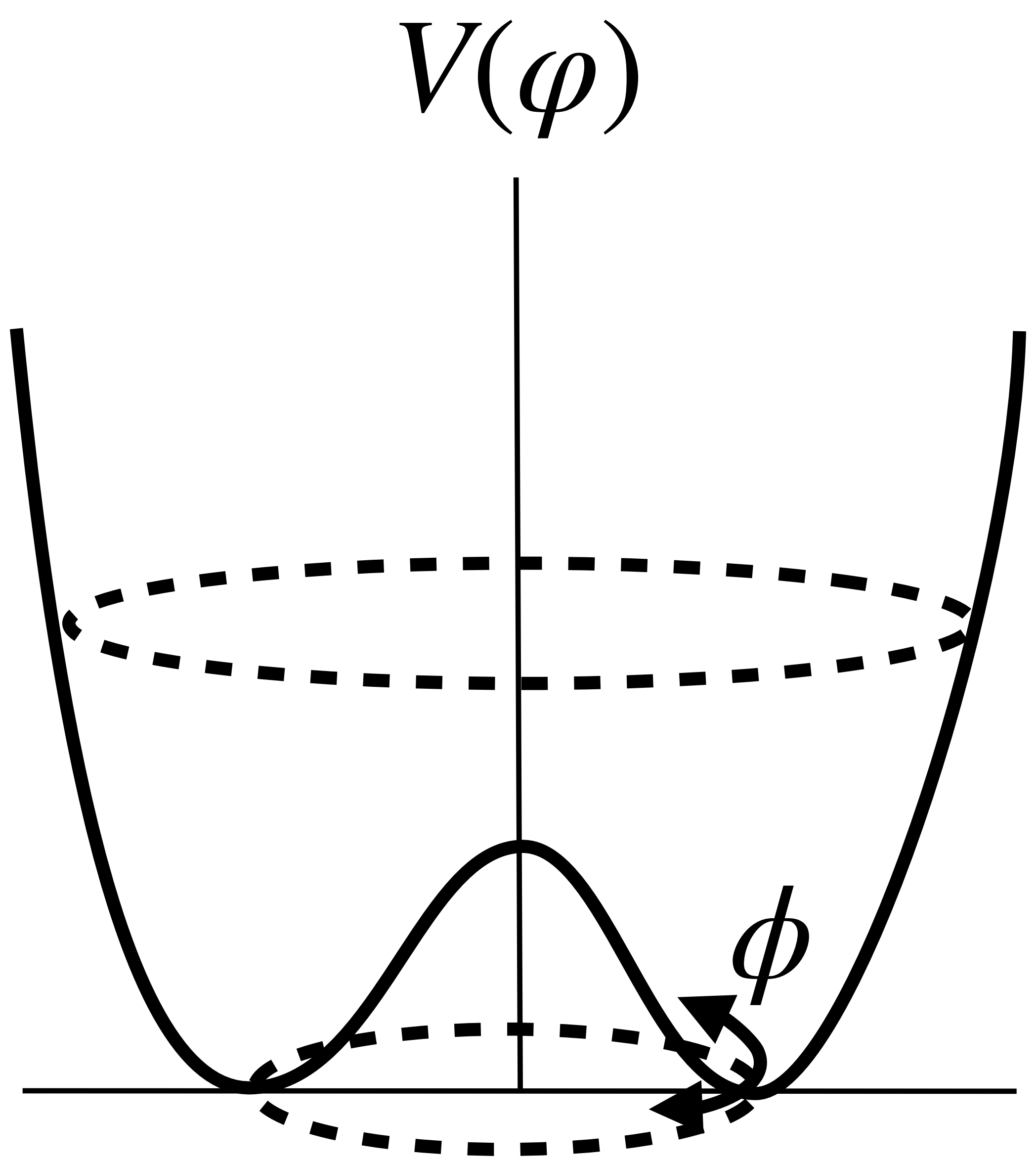


At $t = 0$



1-form symmetry
= flux conservation,
which is broken
if there is a dynamical
electric field
because of screening

Ex) Superfluid



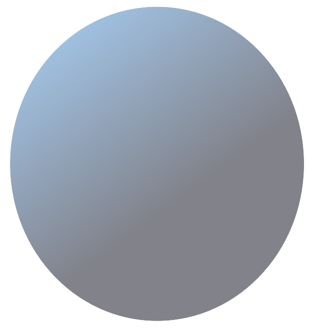
$$S = -\int d^4x \frac{v^2}{2}(\partial_\mu \phi)^2 = -\int \frac{v^2}{2} d\phi \star d\phi$$

Compact scalar: $\phi \sim \phi + 2\pi$

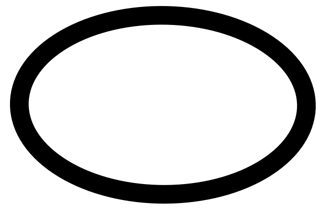
conservation law

$$d \star d\phi = 0 \qquad d(d\phi) = 0$$

$U(1)_E^{[0]} \times U(1)_M^{[2]}$ **symmetries**

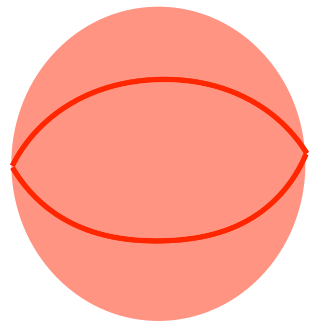


$$U_E = e^{i\theta v^2 \int_V \star d\phi}$$



$$U_M = e^{i\frac{\theta_M}{2\pi}\int_C d\phi}$$

$e^{i\phi}$



V **world surface of vortex**

Non-invertible symmetry
(Categorical symmetry)

Bhardwaj, Tachikawa(2017), Chang, Lin, Shao, Wang, Yin (2018), Ji, Wen (2019),
Komargodski, Ohmori, Roumpedakis, Seifnashri (2020), Nguyen, Tanizaki, Ünsal (2021), Koide, Nagoya, Yamaguchi ('21)

Ex) O(2) gauge theory

cf. Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela, 2104.07036

$$O(2) \simeq U(1) \rtimes \mathbb{Z}_2$$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ **rotation** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **charge conjugation**

Three types of representation, 1, det, 2q

Corresponding Wilson-loops

$$W_{\mathrm{det}}(C) = \mathrm{tr}_{\mathrm{det}} e^{i\int_C a}$$

$$W_{2_q}(C) = e^{iq\int_C a} + e^{-iq\int_C a}$$

Corresponding symmetry generator

$$T_\theta(S) = e^{i\theta \frac{1}{e^2}\int_S \star f} + e^{-i\theta \frac{1}{e^2}\int_S \star f}$$

$$T_\pi(S) = e^{i\pi \frac{1}{e^2}\int_S \star f}$$

These are topological, but not invertible

$$T_\theta(S)T_{\theta'}(S) = T_{\theta+\theta'}(S) + T_{\theta-\theta'}(S)$$

$$T_\theta(S)T_{-\theta}(S) \approx 1 + T_{2\theta}(S)$$

Fusion rule: $T_a(S)T_b(S) = \sum_c N_{ab}{}^c T_c(S)$

cf. Product-to-sum identity

$$2\cos(\theta)\cos(\theta') = \cos(\theta+\theta') + \cos(\theta-\theta')$$

Link between T and W

$$T_\theta(S) = e^{i\theta \frac{1}{e^2}\int_S \star f} + e^{-i\theta \frac{1}{e^2}\int_S \star f}$$

$$W_{2_q}(C) = e^{iq\int_C a} + e^{-iq\int_C a}$$

$$e^{i\theta \frac{1}{e^2}\int_S \star f} e^{iq\int_C a} = e^{iq\theta} e^{iq\int_C a}$$

$$\Rightarrow \quad T_\theta(S) W_{2_q}(C) = (e^{iq\theta} + e^{-iq\theta}) W_{2_q}(C)$$

not a phase

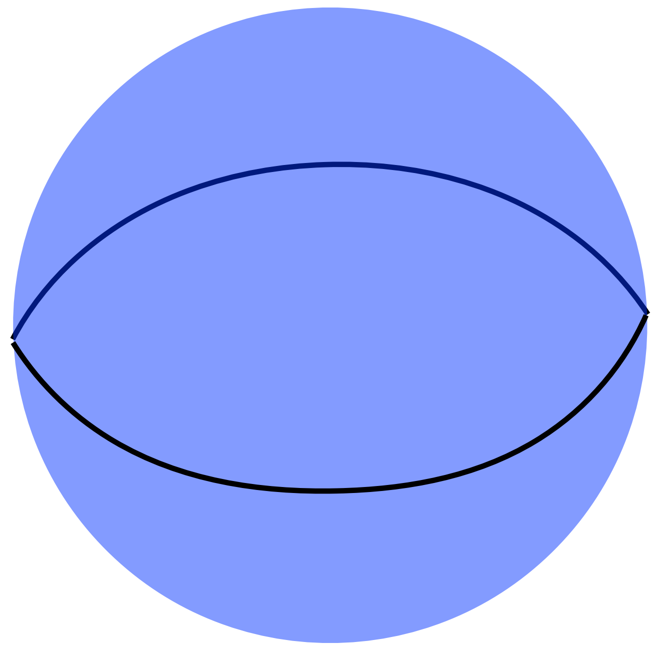
Linking law: $T_\theta(S) W_{2_q}(C) = B_{2_q}(\theta) W_{2_q}(C)$

$$B_{2_q}(\theta) = 2\cos q\theta$$

Noninvertible symmetry in $(d+1)$ dimensions

symmetry generator

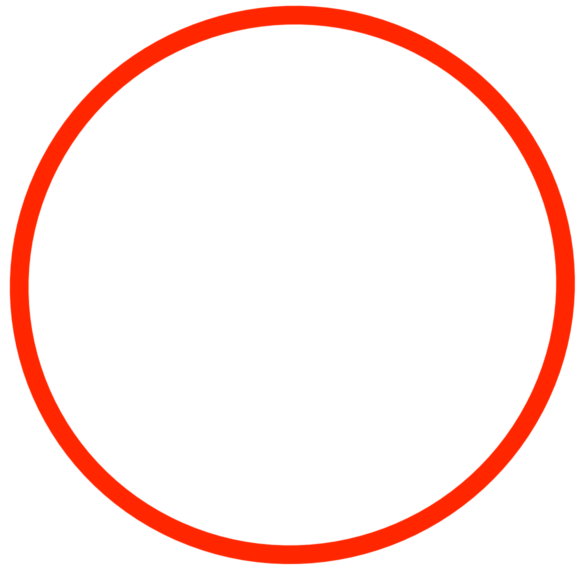
$T_a(M^{d-p})$



$(d-p)$ dimensional topological object labelled by something e.g., a simple object of fusion category

charged object

$W_\rho(C^p)$



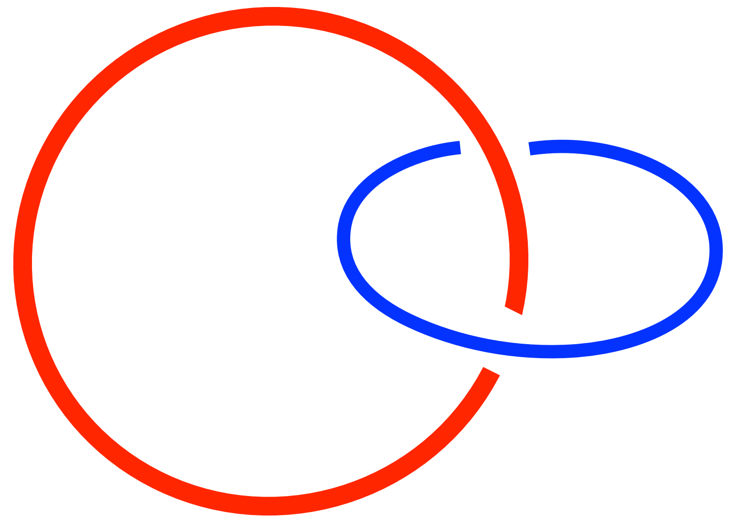
p dimensional object labeled by representation ρ

Noninvertible symmetry in $(d+1)$ dimensions

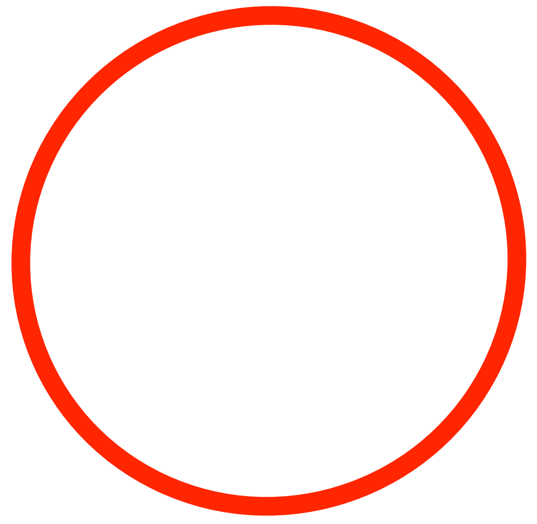
Fusion rule : $T_a(M)T_b(M) = \sum_c N_{ab}{}^c T_c(M)$

Associativity: $T_a(M)(T_b(M)T_c(M)) = (T_a(M)T_b(M))T_c(M)$

Link: $T_a(M)W_\rho(C) = B_\rho(a)W_\rho(C)$



$= B_\rho(a)$



Application

- Spontaneous symmetry breaking
- QCD phase diagram

Nambu-Goldstone Bosons

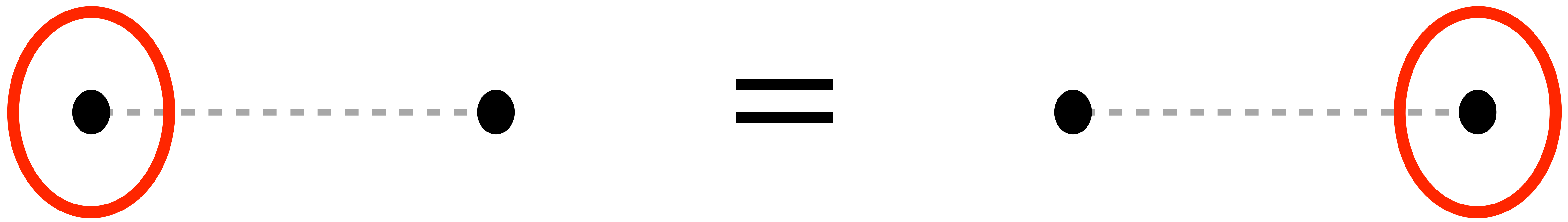
Spontanous symmetry breaking

0 form symmetry

$$\lim_{x\to\infty}\langle\phi^{\dagger}(x)\phi(0)\rangle \simeq \langle\phi^{\dagger}(x)\rangle\langle\phi(0)\rangle \neq 0$$

Off diagonal long range order

two points = boundary of a line



$$\langle e^{i\theta}\phi^{\dagger}(x)\phi(y)\rangle = \langle\phi^{\dagger}(x)e^{i\theta}\phi(y)\rangle$$

long range correlation

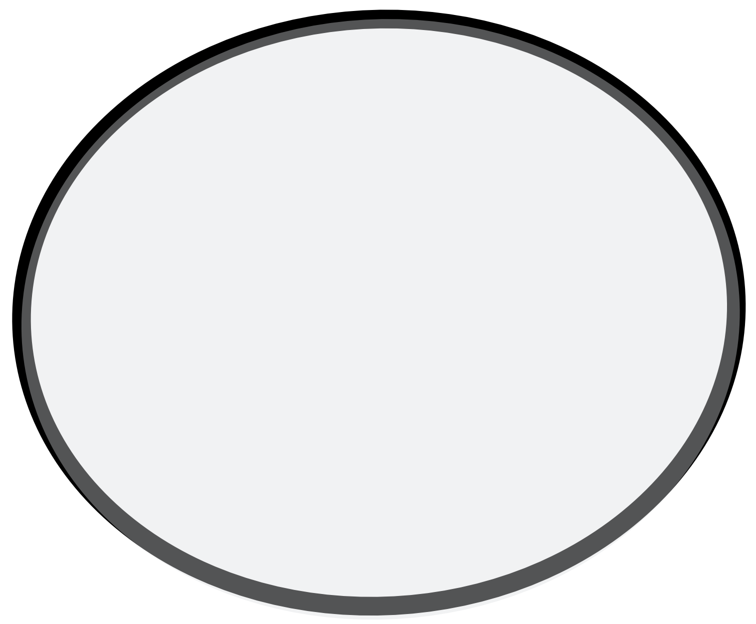
Order parameter

0-form symmetry breaking

$$\lim_{x\to\infty}\langle\phi^\dagger(x)\phi(0)\rangle \neq 0$$

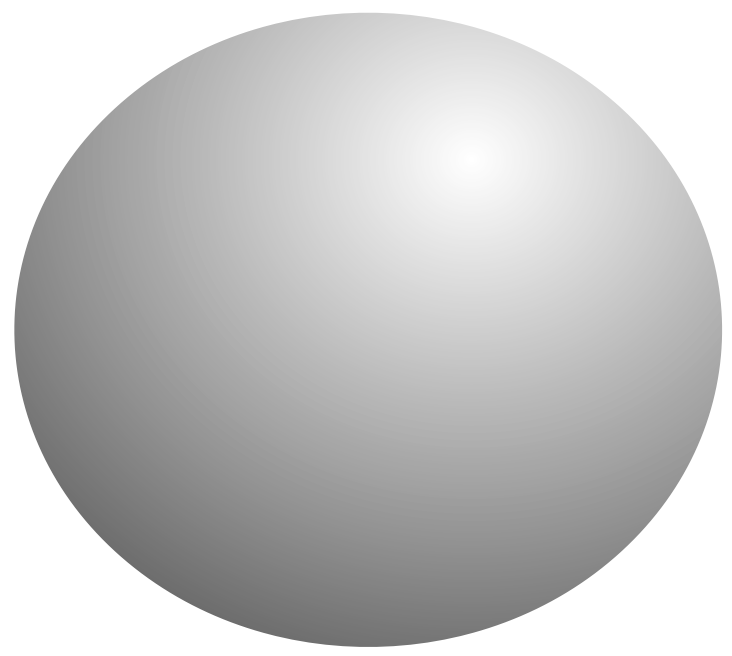
1-form symmetry breaking

$$\lim_{C\to\infty}\langle W(C)\rangle \neq 0$$



p-form symmetry breaking

$$\lim_{M^p\to\infty}\langle W(M^p)\rangle \neq 0$$



Nambu-Goldstone theorem
p form symmetry version

Gaiotto, Kapustin, Seiberg, Willett ('14), Lake ('18), Hofman, Iqbal ('18)

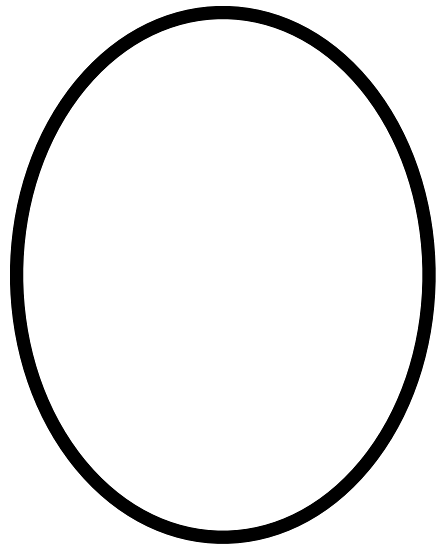
When a continuous p form symmetry is spontaneously broken, a gapless mode appears.

Example) Photons

Gaiotto et al. ('15)

cf. Ferrari, Picasso ('71), Hata ('82), Kugo, Terao, Uehara ('85)

Charged objects

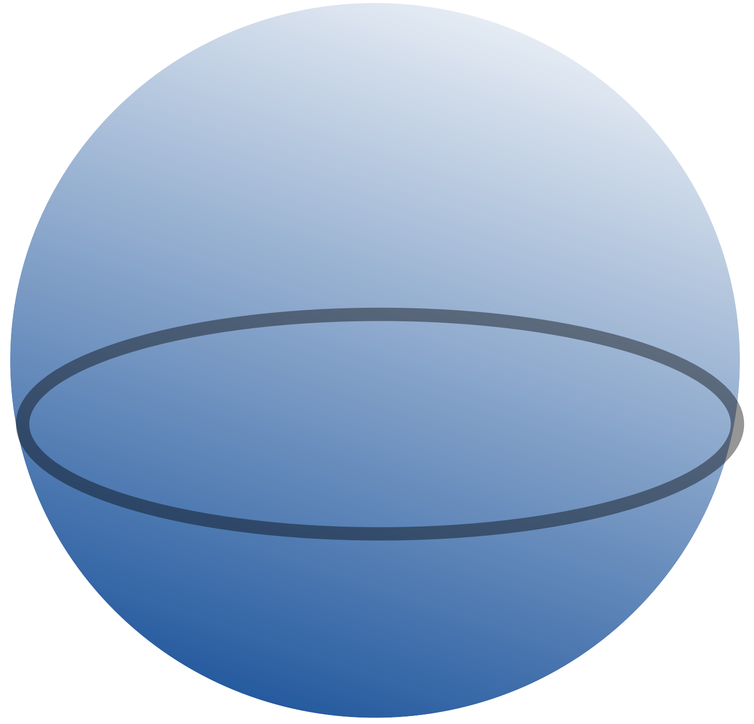


Wilson ('t Hooft) loop

$$W = \exp i \int a$$

$$H = \exp i \int \tilde{a}$$

Symmetry generator



Surface operator

Electric: $U_\theta = \exp \frac{i\theta}{e^2} \int \star f$

Magnetic: $U_\eta = \exp \frac{i\eta}{2\pi} \int f$

Conservation of electric and magnetic flux

Example) Photons

Gaiotto et al. ('15)

cf. Ferrari, Picasso ('71), Hata ('82), Kugo, Terao, Uehara ('85)

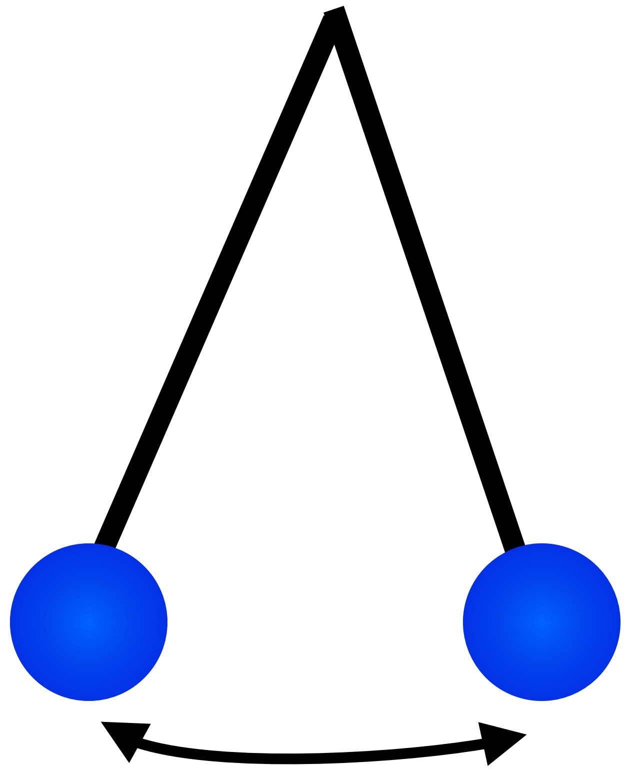
$$\langle \bigcirc \rangle \begin{cases} \text{Area law: unbroken} \\ \text{Coulomb law: broken} \end{cases}$$

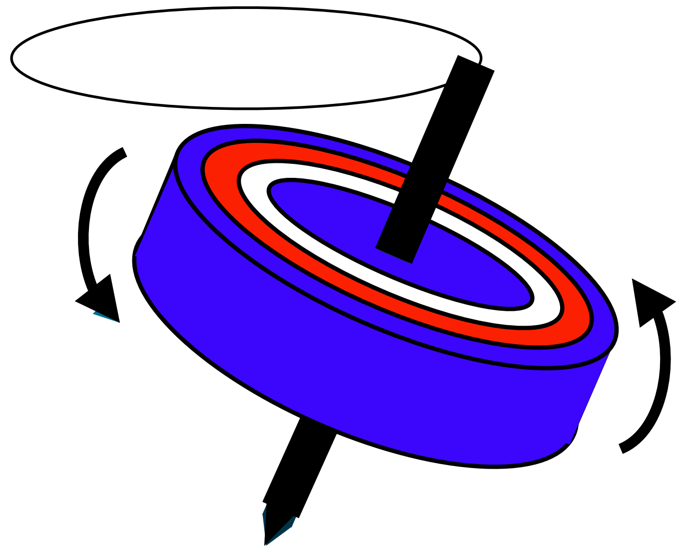
Photons=NG bosons

Three electric fields
but two photons
What is the counting rule?

For 0 form symmetry,
there are two types of NG modes

Watanabe, Murayama ('12)





Type-A
oscillation

Type-B
precession

ex) superfluid phonon

Typically, $\omega \sim k$

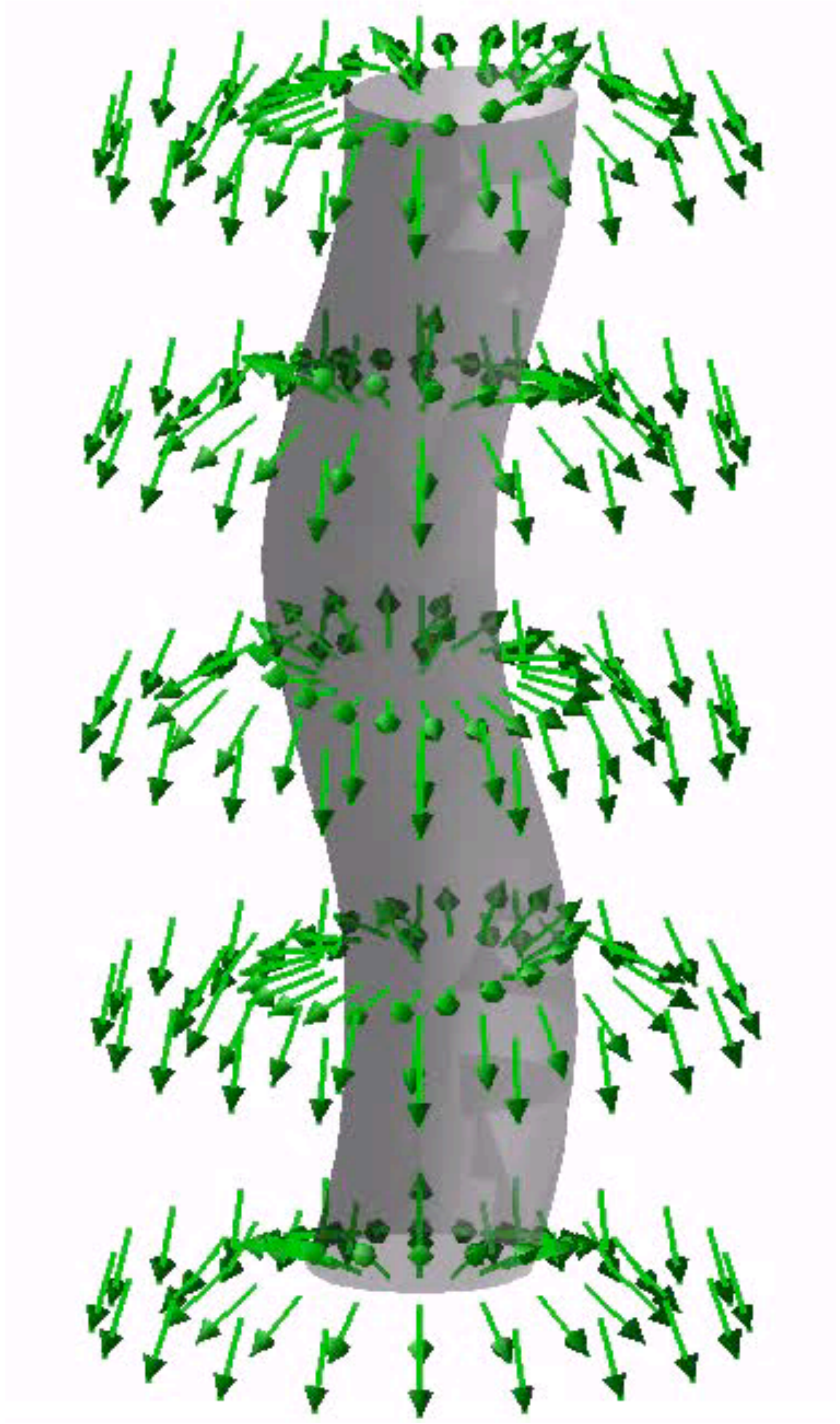
$$N_A = N_{\rm BS} - \mathrm{rank}\langle i[Q_a, Q_b]\rangle$$

ex)magnon

$\omega \sim k^2$

$$N_B = \frac{1}{2}\mathrm{rank}\langle i[Q_a, Q_b]\rangle$$

Ex.) Nonrelativistic $\mathbb{C}P^1$ model

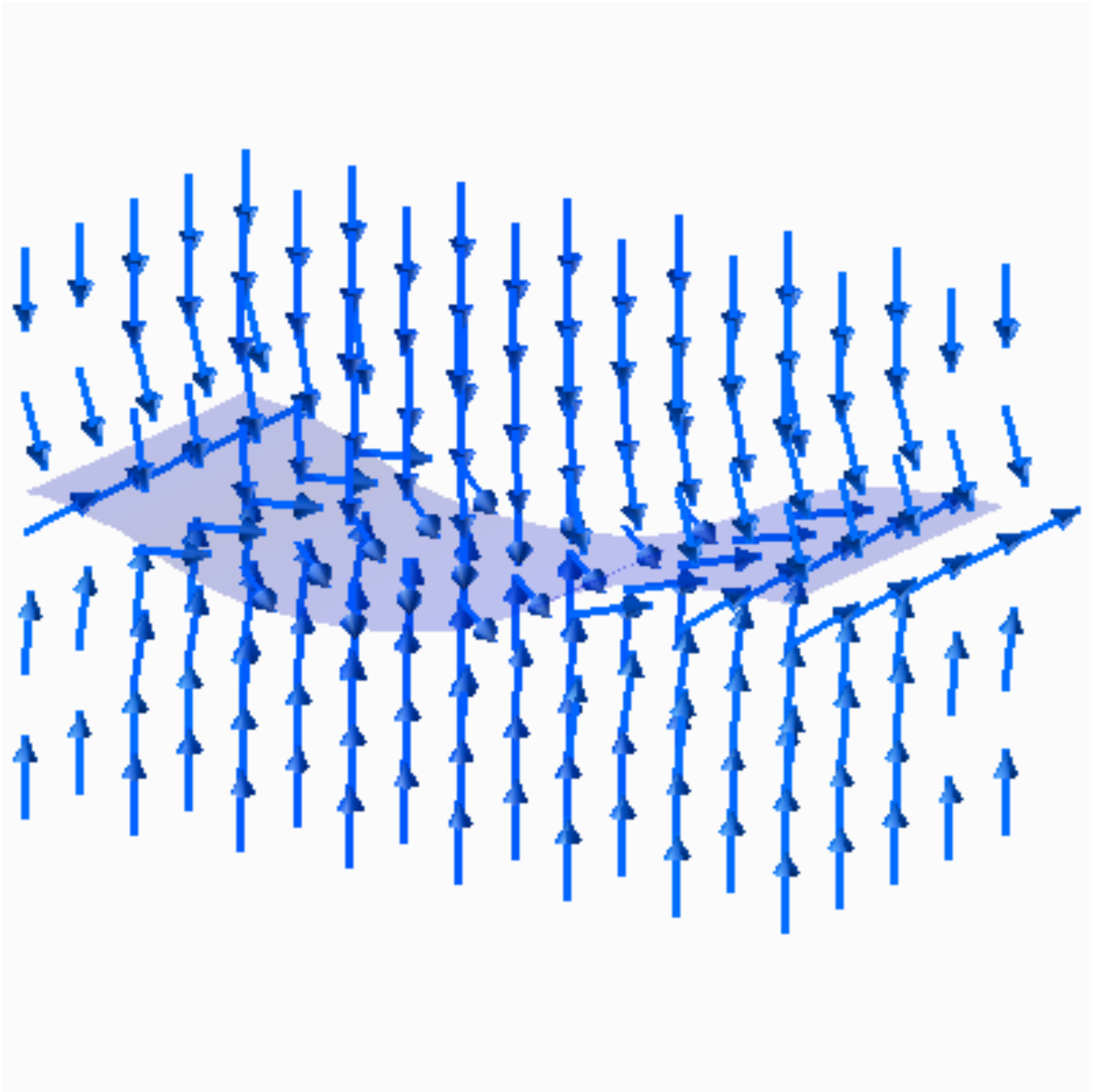


Type-B Kelvon

$$[P_x, P_y] \propto N$$

x translation y translation **1 form symm.**

Kobayashi, Nitta, 1403.4031
c.f. Watanabe, Murayama 1401.8139



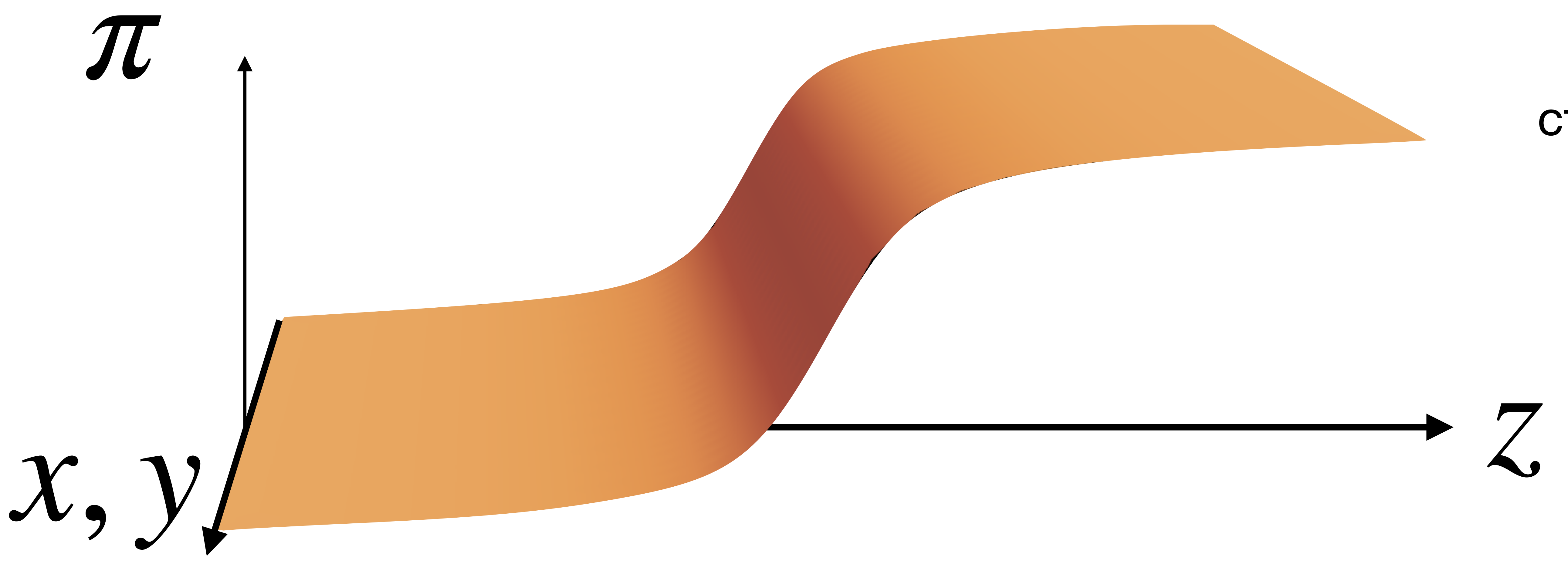
Type-B Ripplon-Magnon

$$[P_z, Q] \propto N$$

z translation **U(1)** **2 form symm.**

Kobayashi, Nitta, 1402.6826

Ex.)Non-relativistic photons



Yamamoto ('15)

cf. Sogabe Yamamoto ('19)

Consider a system of interacting photons and domain walls.

$$S = -\frac{1}{2e^2}\int d^4x\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) + C\int d^4x\pi\boldsymbol{E}\cdot\boldsymbol{B}$$

If $\partial_z\pi = \mathrm{const}$ ➡ $\omega_k \sim k^2$ single photon

$\langle[Q_e(M_{xz}), Q_e(M_{yz})]\rangle \sim \int dz\partial_z\pi$ **Type-B**

Generalization to non-relativistic systems

Y. Hidaka, Y. Hirono, R. Yokokura 2007.15901

Assumptions:

No translational symmetry breaking

Existence of low-energy effective theory describe by Maurer-Cartan form

Method:

Write down possible terms

Counting degrees of freedom using the equations of motion

For 0-form symmetry breaking $G \to H$

DOF: $\xi(P) = e^{i\pi(P)} \in G/H$

"Gauge symmetry ": $\pi(P) \to \pi(P) + 2\pi$

Maurer-Cartan 1 form : $j = \xi^\dagger d\xi$

Effective Lagrangian

$$\mathscr{L} = \mathrm{tr}\,\Omega \wedge j + \mathrm{tr}\, F^2 j \wedge \star j + \cdots$$

p form symmetry

DOF: $W(M) = e^{i\int_M a_A} \in G/H$

M :p_A**dimensional submanifold,** a_A: p_A **form**

Gauge symmetry : $a_A \to a_A + d\lambda^{(p_A-1)}$

Maurer-Cartan (p_A+1) **form** f_A

$$e^{i\int_X f_i} = W(M')^\dagger W(M) \quad \partial X = \bar{M}' \cup M$$

Effective Lagrangian

$$\mathscr{L} = \frac{1}{2} f_A \wedge a_B \Omega^{AB} - \frac{F_{AB}^2}{2} f_A \wedge \star f_B + \cdots$$

Ex:U(1) gauge theory ($\Omega = 0$)

f : E, B six components

Maxwell equations : $df = 0 \quad d \star f = 0$

Two constraint : $\nabla \cdot B = 0 \quad \nabla \cdot E = 0$

6 - 2 = 4 ⇒ 2 modes

In general $D = (d+1)$ dimension, p form:

f : ${}_DC_{p+1}$ **components**

Constraint: ${}_{D-2}C_{p+1}$, ${}_{D-2}C_{p-1}$

# of modes: $\mathcal{N}_{D,p} = \frac{1}{2}({}_DC_{p+1} - {}_{D-2}C_{p+1} - {}_{D-2}C_{p-1}) = {}_{D-2}C_p$

Degree of freedom changes due to $\Omega \neq 0$

Similar to 0-form symmetry,
the first-order derivative is
determined by $\Omega_{AB} \propto \langle [iQ_A, Q_B] \rangle = M_{AB}$.

➡ **# of NG modes change**

For a non relativistic system

Relation between broken symmetry and # of NG modes for 0-form symmetries

Watanabe, Murayama ('12), YH ('12)

$$N_{\mathrm{NG}} = N_{\mathrm{BS}} - \frac{1}{2}\mathrm{rank}\,\langle[\mathrm{i}Q_a, Q_b]\rangle$$

Generalization to higher form symmetries

Hidaka, Hirono, Yokokura ('20)

$$N_{\mathrm{NG}} = \sum_A {}_{d-1}C_{p_A} - \frac{1}{2}\mathrm{rank}\,\langle[\mathrm{i}Q_a, Q_b]\rangle$$

Classification of phases of matter

If the symmetry is different, the phase is different

- **Coulomb phase of QED (vacuum)**

  $U(1)_E^{[1]} \times U(1)_M^{[1]}$ **symmetry**

- **Super conductor**

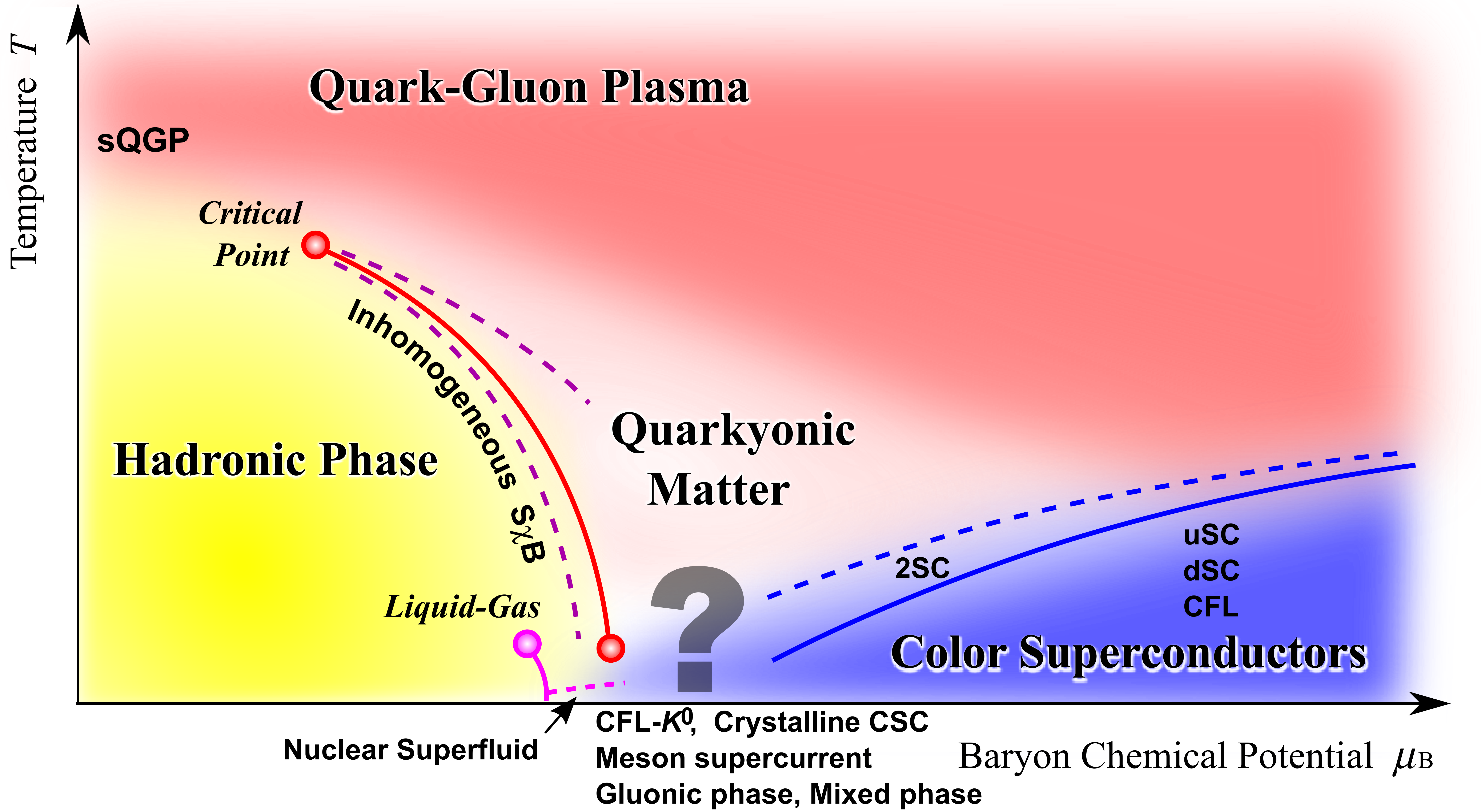
  $\mathbb{Z}_2^{[1]} \times \mathbb{Z}_2^{[2]}$ **symmetry**

- **fractional quantum Hall system**

  $\mathbb{Z}_q^{[1]}$ **symmetry**

QCD phase diagram

Fukushima, Hatsuda, Rept. Prog. Phys. 74 (2011) 014001



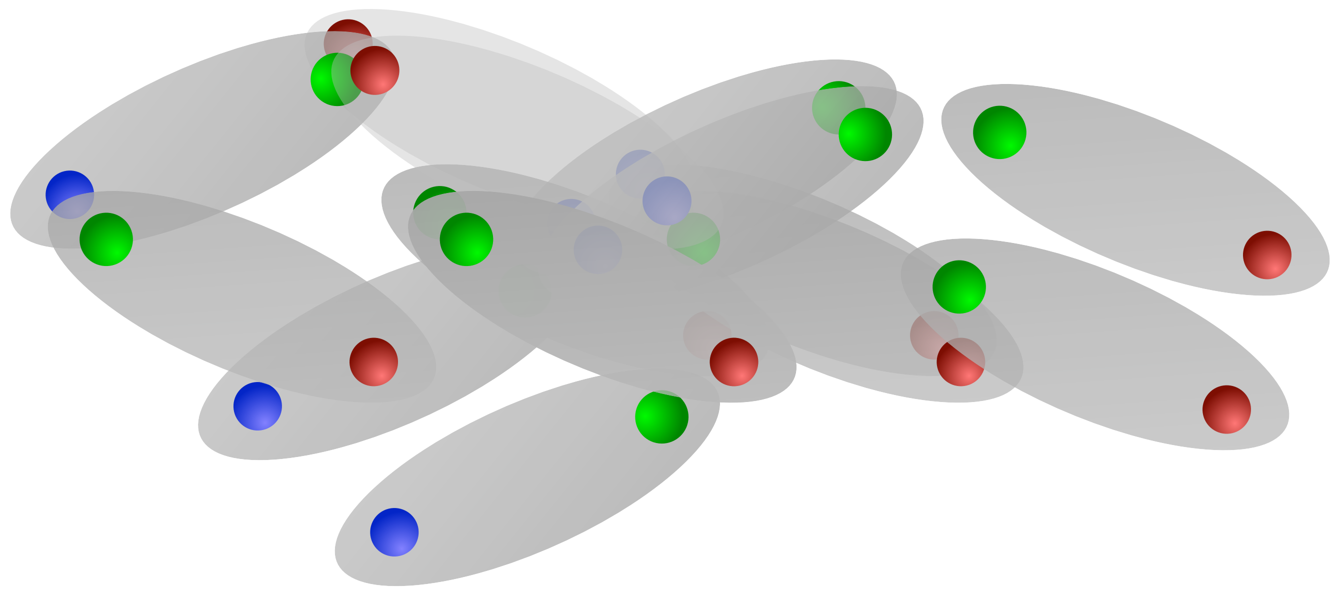
The high-density phase is not well understood.

What we know?

For 3-flavor QCD : $G = SU(3)_L \times SU(3)_R \times U(1)_B$

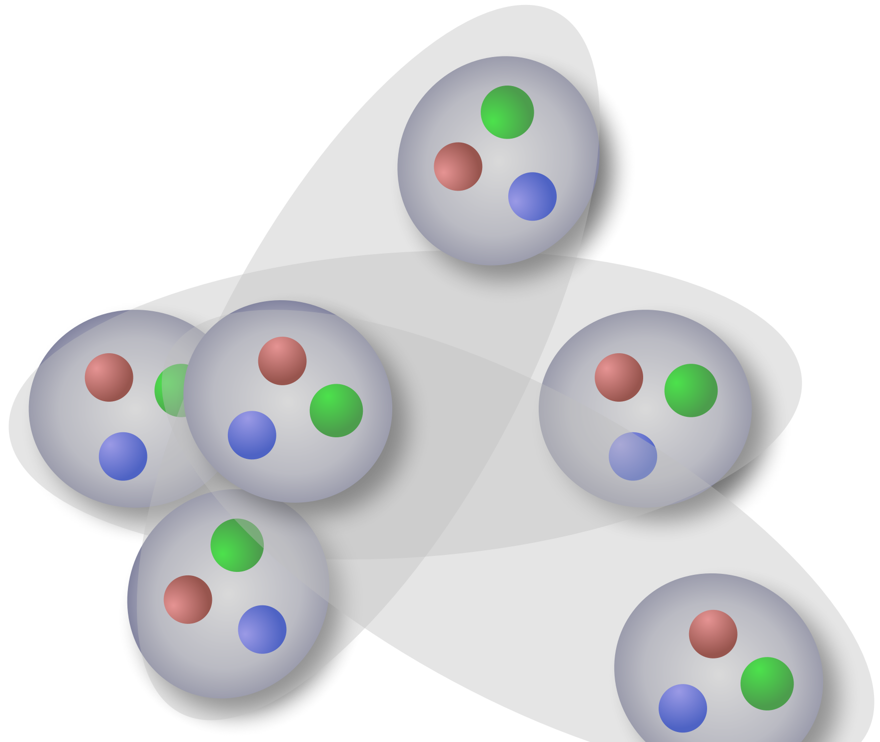
High density:

Color flavor locking phase (CFL phase)



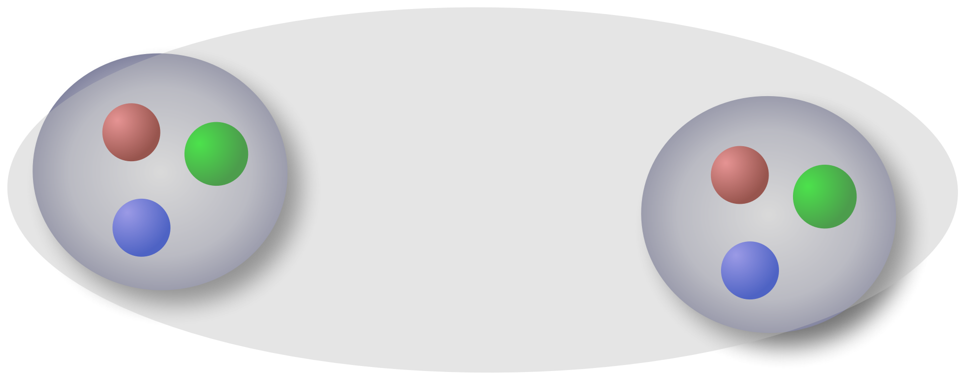
Low density:

Superfluid phase of nuclear matter



Hadronic superfluid phase

di-baryons condense

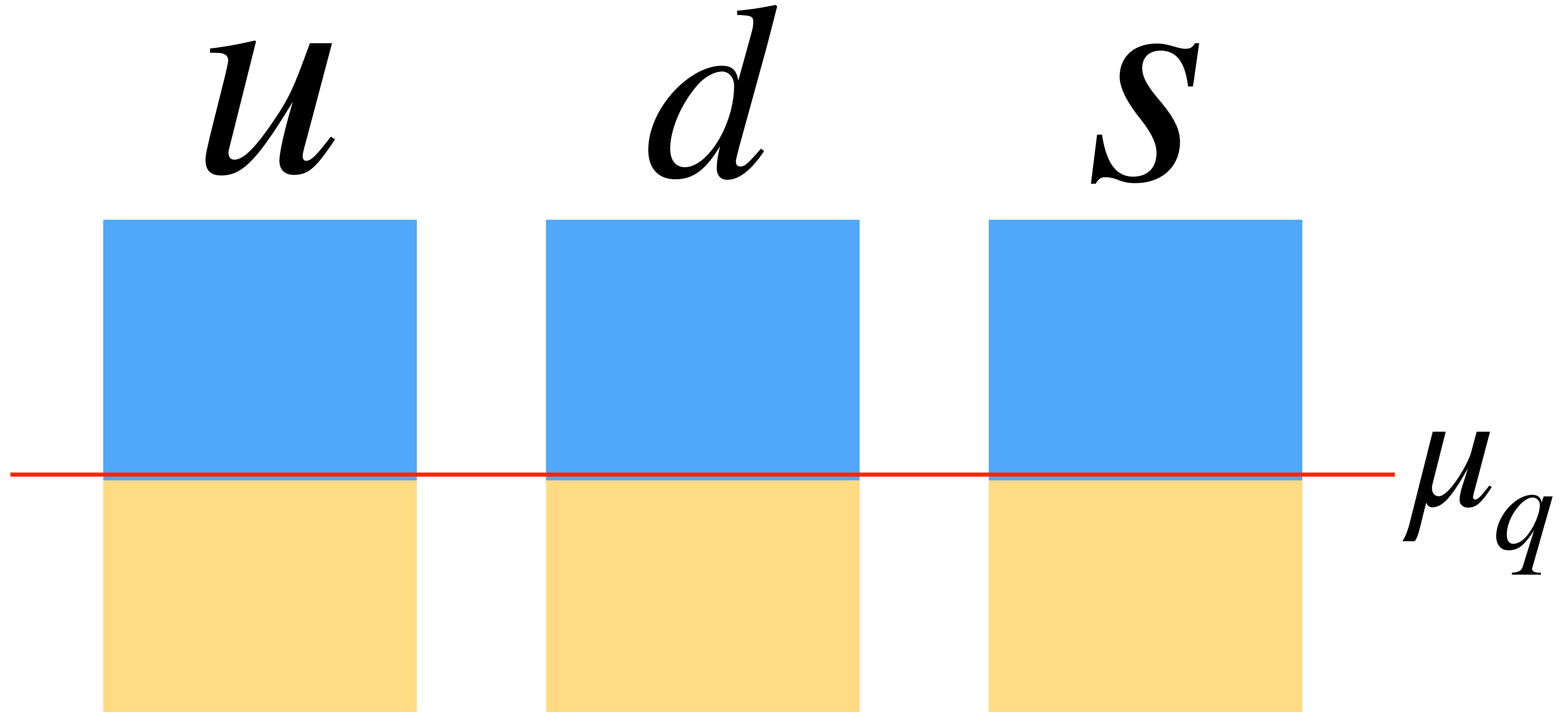


$$\Delta = \langle \Lambda\Lambda \rangle \neq 0 \quad \Lambda \sim uds$$

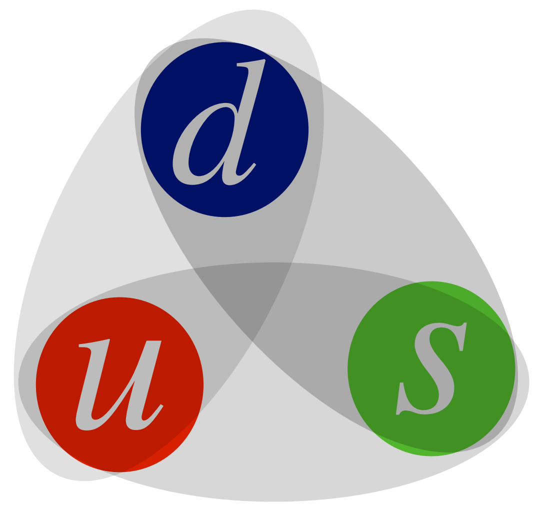
Symmetry breaking pattern

$$SU(3)_L \times SU(3)_R \times U(1)_B \to SU(3)_V$$

CFL phase



quark pair



$$(\Phi_L)^i_a = \epsilon^{ijk}\epsilon_{abc}\langle (q_L)^b_j (Cq_L)^c_k\rangle \quad (\Phi_R)^i_a = \epsilon^{ijk}\epsilon_{abc}\langle (q_R)^b_j (Cq_R)^c_k\rangle$$

$$\Phi := \Phi_L = -\Phi_R = \begin{pmatrix} \Delta_{\mathrm{CFL}} & 0 & 0 \\ 0 & \Delta_{\mathrm{CFL}} & 0 \\ 0 & 0 & \Delta_{\mathrm{CFL}} \end{pmatrix}$$

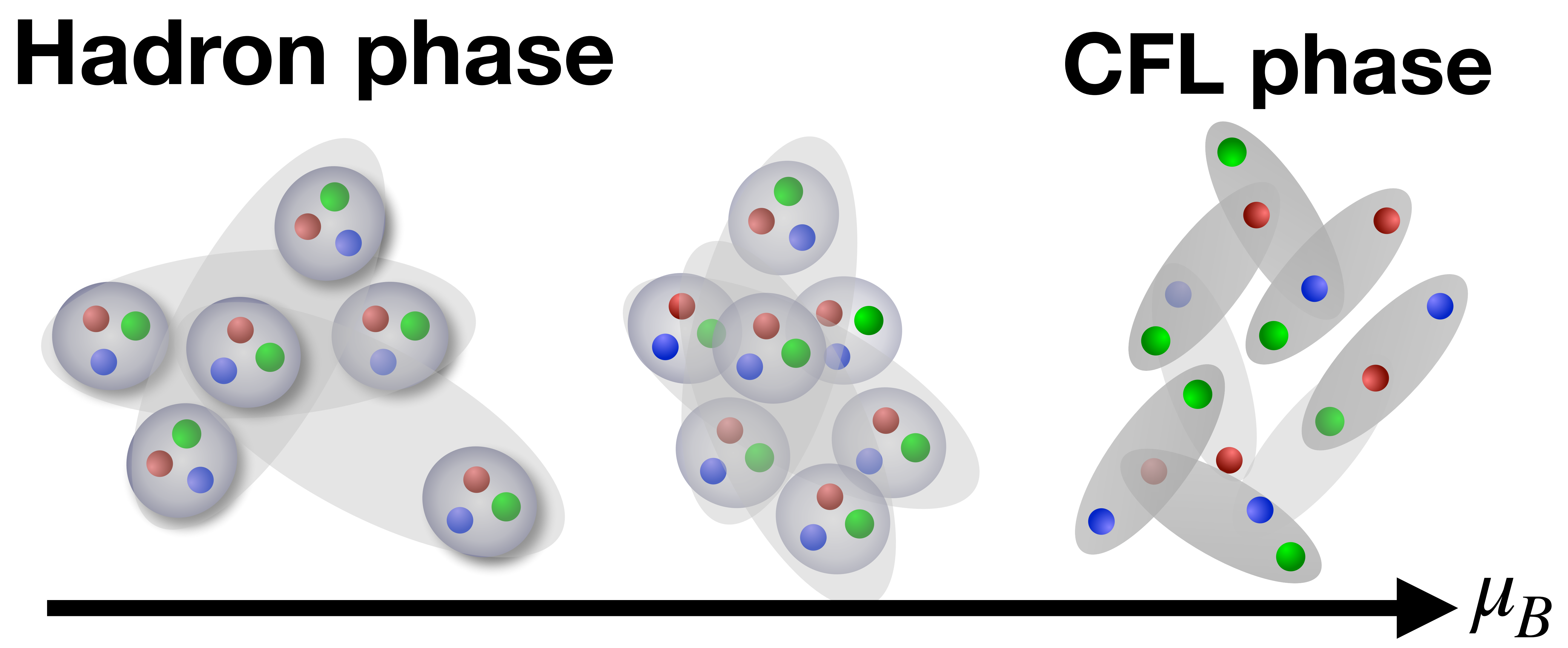
Chiral symmetry breaking

Symmetry breaking pattern of global symmetry

$$SU(3)_L \times SU(3)_R \times U(1)_B \to SU(3)_V$$

What characterizes the CFL phase?

Fradkin-Shenker theorem:
Confined and Higgs phases are the same phase



quark-hadron continuity (hypothesis)

Schafer and Wilczek, PRL 82 , 3956(1999)   Hatsuda, Tachibana, Yamamoto, Baym, PRL 97 122001 (2006)

Topological excitations

cf. Eto, Hirono, Nitta & Yasui, PTEP 2014, 012D01 (2014)

order parameter space $G/H \simeq \frac{SU(3)\times U(1)_B}{\mathbb{Z}_3} \simeq U(3)$

U(1) vortex

$$\Phi := \Delta_{\rm CFL}\begin{pmatrix} e^{i\theta}f(r) & 0 & 0 \\ 0 & e^{i\theta}f(r) & 0 \\ 0 & 0 & e^{i\theta}f(r)\end{pmatrix}$$

Non-abelian vortex

Balachandran, Digal, Matsuura, PRD73, 074009 (2006)

$$\Phi := \Delta_{\rm CFL}\begin{pmatrix} e^{i\theta}f(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & g(r)\end{pmatrix} = \Delta_{\rm CFL}e^{i\frac{\theta}{3}}\begin{pmatrix} e^{i\frac{2\theta}{3}}f(r) & 0 & 0 \\ 0 & e^{-i\frac{\theta}{3}}g(r) & 0 \\ 0 & 0 & e^{-i\frac{\theta}{3}}g(r)\end{pmatrix}$$

$$A_i = -\frac{\epsilon_{ij}x^j}{g_s^2 r^2}(1-h(r))\,{\rm diag}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$$

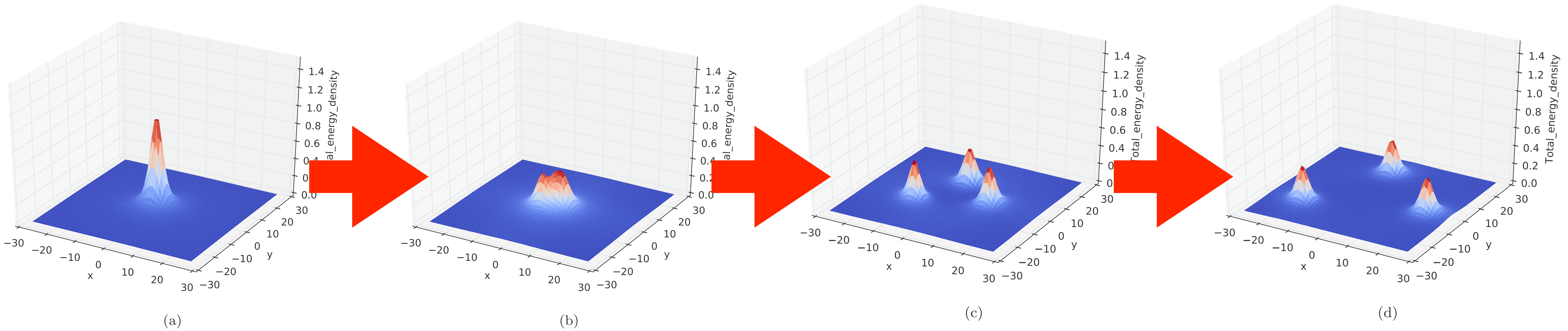
CFL = emergence of $\mathbb{Z}_3$ -2 symmetry

Hirono, Tanizaki, Phys. Rev. Lett. 122, 212001 (2019)

cf. Cherman, Sen, Yaffe, Phys. Rev. D 100, 034015 (2019)

U(1) vortex

non-abelian vortices



Alford, Mallavarapu, Vachaspati, Windisch, PRC 93, 045801 (2016)

If no this symmetry in the hadronic phase
⇒ phase transition

cf. Boojum scenario: Chatterjee, Nitta, Yasui ('19)

Cherman, Jacobson, Sen, Yaffe, Phys. Rev. D 102, 105021 (2020)

cf. Hiono-Tanizaki: unbroken $\mathbb{Z}_3$ -2 form symmetry

⇒ not a topological ordered phase

2 flavor QCD

Hadronic phase :$^3P_2$ superfluid

Dense phase:siglet (ud) +$^3P_2$ (dd) diquark condensate

Fujimoto, Fukushima, Weise Phys. Rev. D 101 (2020) 094009

$$G_{\rm QCD} \supset SU(3)_C \times U(1)_B \xrightarrow{\langle dd \rangle} SO(3) \rtimes (\mathbb{Z}_6)_{C+B} \xrightarrow{\langle ud \rangle} (\mathbb{Z}_3)_{C+B}$$

or $SO(2)_C \times (\mathbb{Z}_6)_{C+B}$

As a vortices "Alice string"

Fujimoto, Nitta, Phys. Rev.D 103 (2021), 114003; 054002; 2103.15185

⇒Emergent 2-form $\mathbb{Z}_3$ or$\mathbb{Z}_6$ symmetry?

or non-invertible symmetry?

If no this symmetry in the hadronic phase
⇒ phase transition

Summary

Symmetry: Topological object labeled with something.

As useful as ordinary symmetry

In particular, useful for classification of gauge theory