Perturbative ambiguities and resurgence in compactified spacetime

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2021/4/20

- K. Ishikawa, O.M., K. Shibata and H. Suzuki, PTEP **2020** (2020) 063B02 [arXiv:2001.07302 [hep-th]]
- O.M. and H. Takaura, PLB **807** (2020) 135570 [arXiv:2003.04759 [hep-th]]
- M. Ashie, O.M., H. Suzuki and H. Takaura, PTEP **2020** (2020) 093B02 [arXiv:2005.07407 [hep-th]]

自己紹介

- 経歴
  - - 2012/3: 名古屋
  - 2012/4 - 2016/3: 九州大学（理学部学士総代）
  - 2016/4 - 2021/3: 九州大学大学院（博士総代+答辞）
- 受賞
  - 令和２年度第１１回育志賞→学振特別研究員(PD)

日本学術振興会 育志賞の概要 —優秀な大学院博士課程学生の顕彰・支援—

1.　趣旨

平成21年、上皇陛下の天皇御即位20年に当たり、日本学術振興会は、社会的に厳しい経済環境の中で、勉学や研究に励んでいる若手研究者を支援・奨励するための事業の資として、上皇陛下から御下賜金を賜りました。
このような陛下のお気持ちを受けて、本会では、将来、我が国の学術研究の発展に寄与することが期待される優秀な大学院博士課程学生を顕彰することで、その勉学及び研究意欲を高め、若手研究者の養成を図ることを目的として、平成22年度に「日本学術振興会 育志賞」を創設しました。

Works

- Gradient flow
  - Lattice formulation of chiral gauge theory
  - Renormalization group
  - Axial anomaly in curved spacetime
  - 4D $\mathcal{N}=2$ SYM supercurrent
- Numerical simulation of 2D $\mathcal{N}=(2,2)$ Landau–Ginzburg model (2020/6 Seminar)
- Resurgence theory (renormalon)
  - Perturbative ambiguities in compactified spacetime

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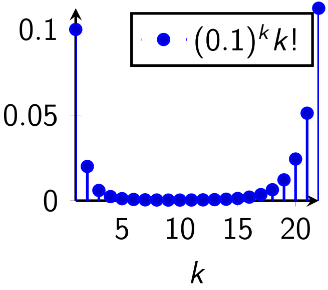
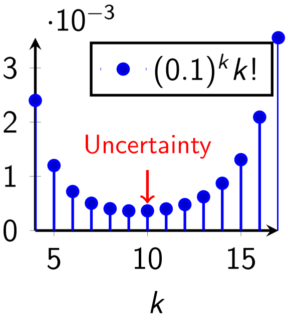
Factorial growth of perturbation series

- In QM/QFT, perturbative expansions of observables are divergent series
- Typically,

$$F(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k, \qquad c_k \sim k! \text{ at large } k.$$





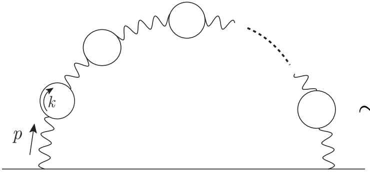
- Accuracy of perturbative predictions is limited. . .

Factorial growth in QFT

- Two sources of factorial growth of perturbative coefficients:
  1. Proliferation of Feynman diagrams (PFD)

	Ground state energy in QM
Zeeman effect	$\sim (-1)^k (2k)!$
Stark effect	$\sim (2k)!$
$V(\phi) \sim \phi^3$	$\sim \Gamma(k+1/2)$
$V(\phi) \sim \phi^4$	$\sim (-1)^k \Gamma(k+1/2)$
Double well	$\sim k!$
Periodic cosine	$\sim k!$

  2. Renormalon ['t Hooft '79]



$\sim \beta_0^k k!$

(β_0: one-loop coefficient of the beta function)

Borel resummation

- Borel resummation: summing divergent asymptotic series

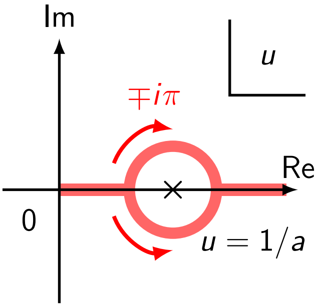
$$f(g^2) \sim \sum_{k=0}^{\infty} f_k \left(\frac{g^2}{16\pi^2}\right)^{k+1} \qquad \text{with } f_k \sim a^k k! \text{ as } k \to \infty$$

$\Downarrow$ Borel transform

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k = \frac{1}{1-au} \qquad \text{(Pole singularity at } u = 1/a\text{)}.$$

- The Borel sum is given by

$$f(g^2) \equiv \int_0^{\infty} du\, B(u) e^{-16\pi^2 u/g^2}.$$



- $a < 0$ (alternating series) $\to$ convergent
- $a > 0 \to$ ill-defined due to the pole

  $\Rightarrow$ Imaginary ambiguity $\sim \pm e^{-16\pi^2/(ag^2)}$

Resurgence theory and renormalon problem

- Perturbative ambiguities (Borel resummation)

$$\text{PFD}: \quad \sum_k k! \left(g^2/16\pi^2\right)^k \qquad \Rightarrow \delta \sim \exp\left(-16\pi^2/g^2\right)$$

$$\text{Renormalon}: \quad \sum_k \beta_0^k k! \left(g^2/16\pi^2\right)^k \qquad \Rightarrow \delta \sim \exp\left[-16\pi^2/(\beta_0 g^2)\right]$$

- Resurgence structure [Bogomolny '80, Zinn-Justin '81]

$$\delta_{\text{PFD}} \sim \exp\left(-16\pi^2/g^2\right) = \exp\left(-2S_I\right)$$

$\Updownarrow$ Cancellation

$$\delta_{\text{Instanton}} \sim \exp\left(-2S_I\right) \qquad \left(\text{Instanton action } S_I = 8\pi^2/g^2\right)$$

- But, what cancels the renormalon ambiguity $\delta_{\text{Renormalon}}$?
  - Non-trivial configuration with $S = S_I/\beta_0$ ($\delta_{\text{NP}} \sim e^{-2S_I/\beta_0}$)?
  - Not known (cf. cancellation between terms of OPE [David '82])
  - Higher-order perturbation theory?

Renormalon cancellation in $\mathbb{R}^{d-1} \times S^1$?

- A possible candidate [Argyres–Ünsal '12, Dunne–Ünsal '12, ...]:
  Bion [Ünsal '07] = fractional instanton/anti-instanton pair
  on $\mathbb{R}^{d-1} \times S^1$ with $\mathbb{Z}_N$-twisted boundary conditions (BC)

$$\underbrace{\varphi^A(\boldsymbol{x}, x_d + 2\pi R)}_{N\text{-component field}} = e^{2\pi i m_A R} \varphi^A(\boldsymbol{x}, x_d), \; m_A R = \begin{cases} A/N & \text{for } A \neq N \\ 0 & \text{for } A = N \end{cases}$$

- $S_B = S_I / N \leftarrow$ similar N dependence! ($\beta_0 = \frac{11}{3} N$ for $SU(N)$ GT)
- Resurgence on S^1 compactified spacetime?

	PT	Semi-classical object
~~$\mathbb{R}^4$~~	$\delta_{\text{Renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$	?
$\mathbb{R}^3 \times S^1$	$\delta_{\text{Renormalon}}$?	$\delta_{\text{Bion}} \sim e^{-2S_B} = e^{-2S_I/N}$

  - QFT on $\mathbb{R}^4 \overset{\text{Def.}}{=}$ QFT on $\mathbb{R}^3 \times S^1$ in $R \to \infty$

Resurgence structure in $\mathbb{R}^{d-1} \times S^1$

- *No renormalons* in $SU(2)$ and $SU(3)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ [Anber–Sulejmanpasic '14]
- Questions:
  1. What cancels the bion ambiguity? (PFD?)
  2. Are there no renormalons in other theories? ($SU({}^\forall N)$?)
  3. How does $\delta_{\text{Renormalon}}$ on $\mathbb{R}^4$ emerge in $R \to \infty$?
- Answers [O.M.–Takaura, Ashie–O.M.–Suzuki–Takaura]:
  1. Enhancement of PFD due to $\mathbb{Z}_N$ twisted BC: $k! \to N^k k!$

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{Bion}} \sim e^{-2S_I/N}$

  2. No renormalons in $SU({}^\forall N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
  3. "Renormalon precursor" $\stackrel{R\to\infty}{\to}$ Renormalon on $\mathbb{R}^4$
- Helpful in giving a unified understanding on resurgence in QFT

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Enhancement of PFD ambiguity

- Enhanced PFD-type ambiguities cancel bion ambiguities

$$\text{PFD: } k! \Rightarrow N^k k!$$

$$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2} \Rightarrow \delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)} \leftrightarrow \delta_{\text{bion}}|_{\mathbb{R}^3\times S^1}$$

- Mechanism ($\sim$ Linde problem ['80] in finite-temperature QFT):
  1. S^1 compactification $\to$ IR divergences

$$\int d^d p \cdots = \text{finite} \to \int d^{d-1}p \frac{1}{2\pi R} \sum_{p_d \in \mathbb{Z}/R} \cdots = \infty \text{ at IR}$$

  2. twisted BC $\to$ twist angles as IR regulators

$$\frac{1}{p^2} \to \frac{1}{\boldsymbol{p}^2 + (p_d + m_A)^2} \xrightarrow[p_d=0, A=1]{\text{Amplitude}} \left(\frac{g^2}{m_A R}\right)^k = \underbrace{(Ng^2)^k}_{\text{Enhancement!}}$$

- Let us study the $\mathbb{C}P^N$ model on $\mathbb{R}\times S^1$
  - Bion calculus [Fujimori-Kamata-Misumi-Nitta-Sakai '18]

2-dimensional $\mathbb{C}P^N$ model with twisted BC

- 2D $\mathbb{C}P^{N-1}$ model with $\mathbb{Z}_N$-twisted BC $(A = 1, 2, \ldots, N)$

$$S = \frac{1}{g^2} \int d^2x \left[\partial_\mu \bar{z}^A \partial_\mu z^A - j_\mu j_\mu + f(\bar{z}^A z^A - 1) \right]$$

where $j_\mu = (1/2i)\bar{z}^A \overleftrightarrow{\partial}_\mu z^A$, f is an auxiliary field $(\bar{z}^A z^A = 1)$

- $\#$ of vacuum bubble diagrams, T_k $(j_\mu \to \bar{z}z$, propagator $\to 1)$

$$T_k = \frac{1}{(2k)!} \left(\frac{\delta}{\delta \bar{z}} \frac{\delta}{\delta z} \right)^{2k} \frac{1}{k!} \left[(\bar{z}z)^2 \right]^k \sim 4^k \Gamma(k + 1/2)$$

- Amplitude of $(k+1)$-loop connected diagram

$$\xrightarrow{p_2=0, A=1} \frac{V_2 (g^2)^k}{(2\pi R)^{k+1}} \int \left(\prod_{i=1}^{k+1} \frac{dp_{i,1}}{2\pi} \right) \frac{F^{2k}(p_{i,1}, m_A)}{\prod_{i=1}^{2k} [q_{i,1}^2 + m_A^2 + f_0]}$$

(F^{2k}: $2k$th-order polynomial, q: linear combination of $\{p_i\}$, $f_0 = \langle f \rangle$)

- IR divergence in massless limit $m_A^2 + f_0 \to 0$ $(\int d^2p \to \int dp)$

IR structure and enhancement mechanism

- $m_A^2 + f_0 = 1/(NR)^2 + f_0$ works as an IR regulator
  - $m_A^2 \gg f_0$

$$\sim \frac{V_2}{R^2} \frac{1}{(m_A R)^{k-1}} \left(\frac{g^2}{4\pi} \right)^k \xrightarrow{A=1} \frac{V_2}{R^2} \frac{1}{N} \left(\frac{N g^2}{4\pi} \right)^k \quad \text{Enhancement!}$$

  - $m_A^2 \ll f_0$

$$\sim \frac{V_2}{R^2} \sum_{\alpha \geq 0} \frac{(m_A R)^\alpha}{(f_0 R^2)^{(k+\alpha-1)/2}} \left(\frac{g^2}{4\pi} \right)^k \quad \text{~~Enhancement~~}$$

- Dependence on $NR\Lambda$ ($\Lambda = \mu e^{-2\pi/(\beta_0 g^2)}$: dynamical scale)
  - $NR\Lambda \ll 1$ (Bion calculus is valid)

$$\sqrt{f_0} R \sim \frac{g^2}{4\pi} \Rightarrow [m_A^2 = \mathcal{O}(g^0)] \gg [f_0 = \mathcal{O}(g^4)] \quad \text{Enhancement!}$$

  - $NR\Lambda \gg 1$ ("Large N")

$$f_0 \sim \Lambda^2 \quad \Rightarrow \quad \frac{m_A^2}{f_0} = \frac{1}{(NR\Lambda)^2} \ll 1 \quad \text{~~Enhancement~~}$$

Discussions on enhancement mechanism

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\mathrm{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\mathrm{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\mathrm{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\mathrm{Bion}} \sim e^{-2S_I/N}$

- Enhancement phenomenon is consistent with bion
- Borel singularities at $m_A R = A/N = \underbrace{1/N}_{\text{1-bion}}, \underbrace{2/N}_{\text{2-bion}}, \ldots$
- Disconnected diagram (n connected) is suppressed by $1/N^{n-1}$
  - $\#$ of connected diagrams $\sim \ln \left[\sum_k T_k (g^2)^k\right]$
- Sum over flavor indices gives rise to further N factor
- Vacuum energy of 2D SUSY $\mathbb{C}P^N$ model (following slides)
  - $NR\Lambda \gg 1$ [Ishikawa–O.M.–Shibata–Suzuki]
  - $NR\Lambda \ll 1$ [Fujimori–Kamata–Misumi–Nitta–Sakai '18]

Vacuum energy of 2D SUSY $\mathbb{C}P^{N-1}$ model

- 2D SUSY $\mathbb{C}P^{N-1}$ model with SUSY breaking term

$$S = \frac{N}{\lambda}\int d^2x \left[-\bar{z}^A D_\mu D_\mu z^A + \bar{\sigma}\sigma + \bar{\chi}^A(\not{D} + \bar{\sigma}P_+ + \sigma P_-)\chi^A\right]$$
$$+ \int d^2x \frac{\delta\epsilon}{\pi R}\sum_A m_A \left(\bar{z}^A z^A - \frac{1}{N}\right)$$

where $D_\mu = \partial_\mu + iA_\mu$, $\gamma_5 = -i\gamma_x\gamma_y$, $P_\pm = (1 \pm \gamma_5)/2$, and we impose $\bar{z}^A z^A = 1$, $\bar{z}^A\chi^A = 0$ and $\bar{z}^A\chi^A = 0$.

- Vacuum energy as a function of $\delta\epsilon$

$$E(\delta\epsilon) = \underbrace{E^{(0)}}_{=0} + E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \dots$$

- Vacuum energy at $NR\Lambda \ll 1$ [Fujimori-Kamata-Misumi-Nitta-Sakai]

$$E_{\text{Bion}} = 2\Lambda\sum_{b=1}^{N-1}(-1)^b\frac{b}{(b!)^2}(NR\Lambda)^{2b-1}$$
$$\times\left\{\delta\epsilon + \left[-2\gamma_E - 2\ln\left(4\pi b/\lambda_R\right) \mp \pi i\right]\delta\epsilon^2 + \dots\right\}$$

Vacuum energy in terms of $1/N$ expansion

- Vacuum energy ($NR\Lambda \gg 1$) [Ishikawa–O.M.–Shibata–Suzuki]

$$E^{(1)}\delta\epsilon = 2\delta\epsilon \sum_A m_A \left\langle \bar{z}^A z^A - 1/N \right\rangle_{\delta\epsilon=0}$$

$$= 0 \cdot N^0 + 0 \cdot N^{-1} + \mathcal{O}(N^{-2})$$

$RE^{(2)}\delta\epsilon^2$

$$= -\frac{\delta\epsilon^2}{\pi} \int d^2x \sum_{A,B} m_A m_B$$

$$\times \left\langle \bar{z}^A z^A(x) \bar{z}^B z^B(0) \right\rangle_{\delta\epsilon=0}$$

$$= N^{-1}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-2} F(\Lambda R)$$

$$+ N^{-2}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-3} G(\Lambda R) + O(N^{-3})$$

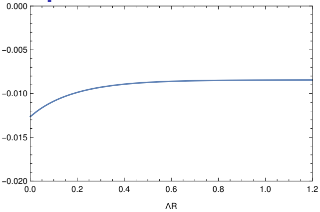

Figure: F

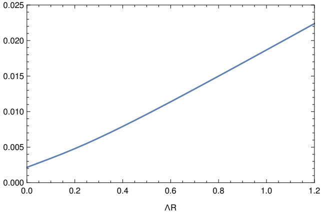

Figure: G

- $RE^{(2)} \to \infty$ as $\Lambda \to 0$; no well-defined weak coupling expansion

N-enhancement versus $1/\Lambda$ (m_A^2 vs f_0)

- Vacuum energy for $NR\Lambda \gg 1$ and $NR\Lambda \ll 1$

$$
\begin{array}{ccc}
E_{\text{Large } N} \sim N^{-1}(\Lambda R)^{-2}\delta\epsilon^2 & & \Rightarrow f_0 = \Lambda^2 \text{ in denominator} \\
\updownarrow & & \Downarrow (m_A^2 \text{ vs } f_0) \\
\operatorname{Im} E_{\text{1-bion}} \sim \pm N(\Lambda R)^2\delta\epsilon^2 & & \Leftarrow \text{enhancement of } N
\end{array}
$$

- cf. ground state energy of SUSY $\mathbb{C}P^1$ QM [Fujimori–Kamata–Misumi–Nitta–Sakai '17]

$$
E = m \sum_k A_k (g^2/m)^k
$$

- "Large N" means $NR\Lambda \gg 1$
  - "$1/N$ expansion" $\equiv 1/(NR\Lambda)$ expansion
  - Higher orders in $1/N$ depend on negative powers of Λ
  - $f_0 = \Lambda^2$ dominates the IR structure

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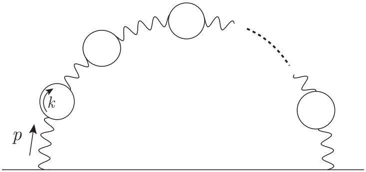
(Definition of) Renormalon ambiguity on $\mathbb{R}^4$

- Renormalon on $\mathbb{R}^4$ can appear from



$$\sim g^2(\mu^2)\sum_k \int \frac{d^4p}{(2\pi)^4}(p^2)^\alpha \Pi(p^2)^k,$$

$$\Pi(p^2) = \beta_0 \frac{g^2(\mu^2)}{16\pi^2} \ln \frac{e^C\mu^2}{p^2}$$

(α, C: constants)

- As $p \to 0$, logarithmic factor in vacuum polarization is crucial

$$\int_p (p^2)^\alpha (\ln p^2)^k \overset{p\sim 0}{\to} k! \quad \overset{\text{Borel}}{\Rightarrow} \quad \pm i\pi \frac{1}{\beta_0}(e^C\Lambda^2)^{\alpha+2}$$

where dynamical scale $\Lambda^2 = \mu^2 e^{-16\pi^2/(\beta_0 g^2)}$

  - Borel transform $\to$ momentum integral

$$B(u) = \int d^4p\,(p^2)^\alpha \left(\frac{e^C\mu^2}{p^2}\right)^u \overset{u\geq\alpha+2}{\longrightarrow} \text{IR divergence}$$

$SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$

- Renormalon analysis for $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ ($N = 2, 3$ [Anber-Sulejmanpasic '14], $\forall N$ [Ashie-O.M.-Suzuki-Takaura '20])
- $SU(N)$ gauge theory with n_W-flavor adjoint Weyl fermions

$$S = -\frac{1}{2g_0^2} \int d^4x \, \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) - 2 \int d^4x \, \mathrm{tr}\, \bar{\psi} \gamma_\mu \left(\partial_\mu \psi + [A_\mu, \psi]\right)$$

- $\mathbb{Z}_N$ twisted BC for adjoint representation

$$\psi(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega \psi(x) \Omega^{-1}$$
$$A_\mu(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega A_\mu(x) \Omega^{-1}$$

where $\Omega = e^{i\pi \frac{N+1}{N}} \mathrm{diag}(e^{-i\frac{2\pi}{N}}, e^{-i\frac{2\pi}{N}2}, \ldots, e^{-i\frac{2\pi}{N}N})$.

  - Equivalently, $e^{2\pi R A_3^{(0)}} = \Omega$ under periodic BC

- $\underbrace{\text{Cartan part}}_{N-1}$ of A_μ: $\underbrace{\text{“photon”}}_{\text{massless } U(1)}$, the others of A_μ: $\underbrace{\text{“W-boson”}}_{m \sim |p_3| \geq 1/(NR)}$
- W-boson cannot give rise to renormalon; consider $U(1)^{N-1}$

1-loop effective action

- 1-loop effective action of photon, A_μ^ℓ ($\ell = 1, 2, \ldots, N-1$)

$$\frac{1}{2g^2}\int d^4x d^4y\, A_\mu^\ell(x) A_\nu^r(y) \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)} \times \left[p^2 \mathcal{P}_{\mu\nu}^L(\delta_{\ell r} - L_{\ell r}) + p^2 \mathcal{P}_{\mu\nu}^T(\delta_{\ell r} - T_{\ell r}) + \delta_{\ell r}\xi p_\mu p_\nu\right]$$

  where projection operators, $\mathcal{P}^L$ in S_1, $\mathcal{P}^T$ in $\mathbb{R}^3$
- Vacuum polarization

$$L_{\ell r} = \frac{\beta_0 g^2}{16\pi^2}\left[\delta_{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + \underbrace{f_0(p^2R^2)_{\ell r} - f_2(p^2R^2)_{\ell r}}_{\text{finite volume}}\right],$$

$$T_{\ell r} = \frac{\beta_0 g^2}{16\pi^2}\left[\delta_{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + \underbrace{f_0(p^2R^2)_{\ell r}}_{\text{finite volume}}\right]$$

  - Large-β_0 approximation [Beneke–Braun '94]: Radiative corrections of A_μ are included by $-\frac{2}{3}n_W \to \beta_0 = (\frac{11}{3} - \frac{2}{3}n_W)N$
- If $\Pi(p^2 \to 0) \sim \ln p^2$, there exists the renormalon.

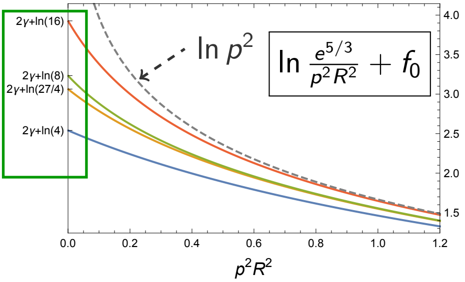
Absence of renormalons on $\mathbb{R}^3 \times S^1$

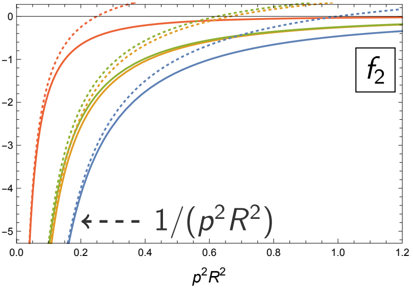
- Asymptotic behavior of L and T in $SU(^\forall N)$ QCD(adj.)

$$L_{\ell r}(p^2R^2) = -\frac{m_{\rm sc}^2}{p^2}\delta_{\ell r} + \text{const.} + \mathcal{O}(p^2R^2\ln(p^2R^2)),$$

$$T_{\ell r}(p^2R^2) = \text{const.} + \mathcal{O}(p^2R^2\ln(p^2R^2)), \qquad m_{\rm sc}^2 \propto \frac{\beta_0 g^2}{16\pi^2 R^2}$$

  - $N = 2\&3$: identical to [Anber-Sulejmanpasic] ($\beta_0 \leftrightarrow \frac{2}{3}(1 - n_W)$)





- No logarithmic factor → no renormalons (no factorial growth)

Backup: Twisted BC and volume independence

- Identity for KK sum with twisted BC:

$$\frac{1}{2\pi R}\sum_{p_d\in\mathbb{Z}/R} F(\boldsymbol{p}, p_d + m_A) = \sum_{n\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{ip_d 2\pi Rn}}F(\boldsymbol{p}, p_d+m_A)$$

$$= \sum_{n\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{i(p_d-m_A)2\pi Rn}}F(\boldsymbol{p}, p_d)$$

- Noting that

$$\sum_A e^{2\pi i m_A Rn}\times 1 = \begin{cases} N, & \text{for } n = 0 \bmod N \text{ (leading)},\\ 0, & \text{for } n \neq 0 \bmod N,\end{cases}$$

the effective radius becomes NR:

$$\xrightarrow{\text{sum over } A} N\sum_{n'\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{ip_d 2\pi NRn'}}F(\boldsymbol{p}, p_d) = \frac{N}{2\pi R}\sum_{p_d\in\mathbb{Z}/(NR)} F(\boldsymbol{p}, p_d)$$

- Assume "large N limit" as $NR\Lambda \to \infty \Rightarrow$ Decompactification
- Volume *dependent*: $\underbrace{\text{other } m_A \text{ dependence}}_{E \text{ in SUSY } \mathbb{C}P^N}$, $\underbrace{\text{~~twisted~~ loop momenta}}_{e.g., U(1) \text{ gauge field}}$

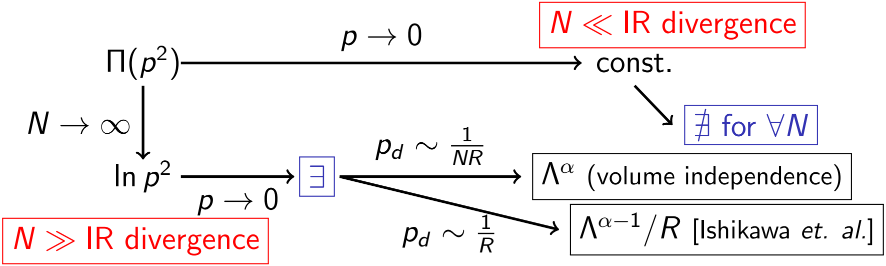
Backup: Subtlety of large N limit on $\mathbb{R}^3 \times S^1$

- Subtlety of naive (apparent) $1/N$ expansion: e.g.,

$$|f_0| \lesssim \frac{1}{N}\frac{1}{(p^2R^2)^{3/2}} \quad \Rightarrow \quad \ln p^2 + f_0(p^2) \overset{\text{finite } p}{\sim} \ln p^2 + \mathcal{O}(1/N)$$

$$\Pi(p^2) \overset{N\to\infty}{\to} \ln p^2;\ {}^{\exists}\text{renormalons [Ashie-O.M.-Suzuki-Takaura-Takeuchi]}$$

- Does there exist renormalons in "$N \to \infty$"?



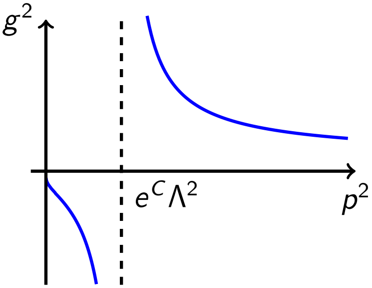
- Volume independence: "$N \gg \infty$" and twisted loop momentum
- Definition of "renormalon" $\overset{\text{Incompatible?}}{\longleftrightarrow}$ $1/N$ expansion

Renormalon from momentum integration on $\mathbb{R}^4$

- $\delta_{\text{renormalon}}\big|_{\mathbb{R}^3\times S^1} = 0$ for $\forall R$, but $\delta_{\text{renormalon}}\big|_{\mathbb{R}^4} = e^{-16\pi^2/(\beta_0 g^2)}$
- How does renormalon on $\mathbb{R}^4$ emerge under $R \to \infty$?
- Reconsider renormalon diagram; resumming the geometric series,

$$g^2(\mu^2)\int \frac{d^4p}{(2\pi)^4}\frac{(p^2)^\alpha}{1-\Pi(p^2)}$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{(p^2)^\alpha}{\frac{\beta_0}{16\pi^2}\ln\frac{p^2}{e^C\Lambda^2}} = \int \frac{d^4p}{(2\pi)^4}(p^2)^\alpha g^2(p^2 e^{-C})$$

- Pole singularity at $p^2 = e^C\Lambda^2$
  $\to$ Contour deformation in $p^2 \in \mathbb{C}$
  $\to$ Renormalon $\sim \pm i\pi\frac{1}{\beta_0}(e^C\Lambda^2)^{\alpha+2}$
  - On $\mathbb{R}^4$, ambiguity of $\int d^4p$
    $=$ ambiguity in Borel sum



"Renormalon precursor" and decompactification

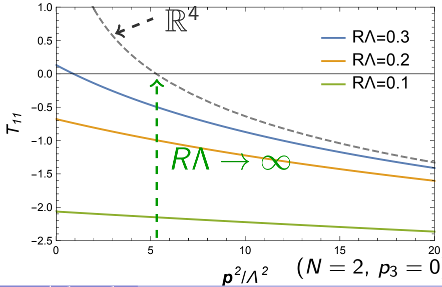
- In $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$,

$$g^2(\mu^2) \int \frac{d^3p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{(p^2)^\alpha}{1 - \Pi(p^2)}$$

$$= \int \frac{d^3p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{\frac{16\pi^2}{\beta_0}(p^2)^\alpha}{\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}}}$$

- "Renormalon precursor" comes from $\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}} = 0$
  - For $p_3 \lesssim \Lambda$, it can exist (no Borel singularities)
  - $\boxed{R \to \infty}$ $f_{\text{finite}} \to 0$ $(\text{pole}|_{\mathbb{R}^3 \times S^1} \to \text{pole}|_{\mathbb{R}^4})$ and $\sum_{p_3} \to \int dp_3$

    Renormalon on $\mathbb{R}^4$ emerge!



$(N = 2,\ p_3 = 0)$

Backup: toy model (large N volume independence)

- Difficult to compute the renormalon precursor directly
- A simplification by large N volume independence: For $|p_3| < e^{5/6}\Lambda$, integrand possesses a simple pole

$$\overset{p^2 \sim e^{5/3}\Lambda^2}{\rightarrow} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{|p_3|<e^{5/6}\Lambda} \frac{16\pi^2}{\beta_0} (p^2)^\alpha \frac{e^{5/3}\Lambda^2}{p^2 - e^{5/3}\Lambda^2}$$

$$\sim \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \frac{1}{e^{5/6}R\Lambda} \sum_{|n|<e^{5/6}R\Lambda} \sqrt{1 - \frac{n^2}{(e^{5/6}R\Lambda)^2}}$$

- Decompactification $R\Lambda \to \infty$

$$\to \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \int_{-1}^{1} dx\, \sqrt{1 - x^2} = \text{Renormalon on } \mathbb{R}^4$$

- $[1 - \Pi(p \to 0)] \times [1 - \Pi(p \to \infty)] < 0$: $^\exists$Renormalon precursor

Contents

Summary

- (Our) Current understanding on resurgence structure in QFT

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{instanton}} \sim e^{-2S_I}$
$\mathbb{R}^4$	$\delta_{\text{renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$	?
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(N g^2)}$	$\delta_{\text{bion}} \sim e^{-2S_I/N}$
$\mathbb{R}^3 \times S^1$	$\delta_{\text{renormalon}} = 0$	No

- Renormalon precursor smoothly reduces renormalon in $R \to \infty$
  - Not associated with Borel singularities
  - Analytic continuation of geometric series
- There remains renormalon puzzle...
- Helpful in giving a unified understanding on resurgence