# Gradient flow exact renormalization group

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- Hidenori Sonoda and H.S., "Derivation of a gradient flow from the exact renormalization group," PTEP **2019**, no.3, 033B05 (2019) [arXiv:1901.05169 [hep-th]].
- Hidenori Sonoda and H.S., "Gradient flow exact renormalization group," PTEP **2021**, no.2, 023B05 (2021) [arXiv:2012.03568 [hep-th]].
- Yuki Miyakawa and H.S., work in progress

# Wilson's exact renormalization group (ERG)

- Dictates change of the system under change of the scale: $S_{\tau=0} \to S_\tau$:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle_{S_\tau} \sim e^{\tau n(D-2)/2} Z_\tau^{n/2} \langle\phi(e^\tau x_1)\cdots\phi(e^\tau x_n)\rangle_{S_{\tau=0}}.$$

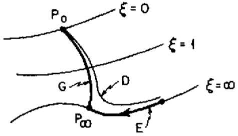



Fig. 12.6. Renormalization group trajectory

- Provides a unique method to construct a quantum field theory, even beyond perturbation theory.
- Nonperturbative fixed point in $D = 4$?
- Non-abelian gauge theory with many flavors, e.g., $G = SU(3)$ with $N_f = 12$; large mass anomalous dimension → technicolor scenario?
- Gravity; asymptotic safety and renormalizable gravity?
- For these theories relevant to particle physicists, gauge symmetry (incl. general coordinate inv.) should be crucial!

# Explicit realization of ERG: Polchinski equation

- Define the partition function with cutoff function $K(p)$:

$$Z = \int [d\phi] \exp\left[-\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0)}\phi(p)\phi(-p) + S_{I,\mathrm{bare}}[\phi]\right], \quad K(p) = \begin{cases} 1 & |p| \to 0, \\ 0 & |p| \to \infty. \end{cases}$$

- $Z$ can be written as

$$Z = \int [d\phi_\ell][d\phi_h] \exp\Bigg[-\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda)}\phi_\ell(p)\phi_\ell(-p) - \frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0) - K(p/\Lambda)} \underbrace{\phi_h(p)\phi_h(-p)}_{\text{contributes only when } \Lambda < |p| < \Lambda_0} + S_{I,\mathrm{bare}}[\phi_\ell + \phi_h]\Bigg].$$

- Integrate over $\phi_h$ (coarse graining) to obtain the Wilson action for the low-frequency mode $\phi_\ell$:

$$Z = \int [d\phi_\ell]\, e^{S_\Lambda[\phi_\ell]}.$$

- Taking the derivative w.r.t. $\Lambda$ ($\Delta \equiv -2p^2 dK/dp^2$),

$$-\Lambda\frac{\partial}{\partial\Lambda} e^{S_\Lambda[\phi]} = \left[\int_p \frac{\Delta(p/\Lambda)}{K(p/\Lambda)}\phi(p)\frac{\delta}{\delta\phi(p)} + \int_p \frac{\Delta(p/\Lambda)}{p^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right] e^{S_\Lambda[\phi]}.$$

# Polchinski equation

- In terms of dimensionless variables (i.e., $\Lambda \to 1$),

$$p \to \Lambda p, \qquad \phi(p) \to \Lambda^{(D-2)/2}\phi(p), \qquad \Lambda = e^{-\tau},$$

$$\begin{aligned}
&\frac{\partial}{\partial\tau} e^{S_\tau[\phi]} \\
&= \int_p \Biggl( \left[ \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \frac{\eta_\tau}{2} + p \cdot \frac{\partial}{\partial p} \right] \phi(p) \cdot \frac{\delta}{\delta\phi(p)} \\
&\qquad + \frac{1}{p^2} \left[ 2\frac{\Delta(p)}{K(p)} k(p) + 2p^2 \frac{dk(p)}{dp^2} - \eta_\tau k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \Biggr) e^{S_\tau[\phi]},
\end{aligned}$$

  where we have generalized the Polchinski equation as $K(p)[1 - K(p)] \to k(p)$ and introducing the anomalous dimension $\eta_\tau$.
- This works fine for scalar field theory, but cannot preserve the gauge symmetry, because

$$\phi(p) \to \phi(p) - \int_q \omega^a(p-q) T^a \phi(q),$$

$$A^a_\mu(p) \to A^a_\mu(p) + ip_\mu \omega^a(p) - \int_q f^{abc} \omega^b(p-q) A^c_\mu(q).$$

- The gradient flow, a gauge covariant diffusion ($\simeq$ coarse graining), provides any cure to this issue?

- Gradient flow:

$$\frac{\partial}{\partial t} B^a_\mu(t,x) = D_\nu \overbrace{G^a_{\nu\mu}}^{\text{field strength of } B}(t,x) = \partial^2 B^a_\mu(t,x) + \text{nonlinear terms}, \quad B^a_\mu(t=0,x) = A^a_\mu(x),$$

- Any correlation function of the diffused gauge field $B^a_\mu$ becomes finite without the wave function renormalization (Lüscher, Weisz) $\Rightarrow$ applications in lattice gauge theory.
- Gradient flow defines the diffusion ($\simeq$ coarse graining) as a function of $t$; much reminiscent of Wilson's ERG:
  - *Nonperturbative gauge coupling* (Lüscher)
  - *"Effective" lattice action* (Kagimura, Tomiya, Yamamura; Yamamura)
  - *3D scalar field theory* (Capponi, Rago, Del Debbio, Ehret, Pellegrini)
  - *Holographic RG* (Aoki, Balog, Onogi, Weisz; Aoki, Yokoyama)
  - *Flow in coupling constant space* (Makino, Morikawa, H.S.)
  - *ERG inspired gradient flow* (Abe, Fukuma)
  - *Stochastic RG* (Carosso, Hasenfratz, Neil; Carosso)
  - *Generalized diffusion equation* (Matsumoto, Tanaka, Tsuchiya)
- Can we identify ERG and GF in some way?
- Any advantage in such identification?
  - Understanding of the finiteness?
  - Gauge invariant ERG?

# Gradient flow (GF) in scalar field theory

- First, consider the scalar field theory; flow equation would be

$$\frac{\partial}{\partial t}\varphi(t,x)=\partial^2\varphi(t,x),\qquad \varphi(t=0,x)=\phi(x).$$

This linear diffusion equation is required by the renormalizability (Capponi, Rago, Del Debbio, Ehret, Pellegrini).

- This flow is trivially solved as

$$\varphi(t,x)=\int d^Dy\int_p e^{ip(x-y)}e^{-tp^2}\phi(y)\quad\text{or}\quad \varphi(t,p)=e^{-tp^2}\phi(p).$$

- In scalar field theory, the diffused field $\varphi$ requires the wave function renormalization as

$$Z_\varphi^{n/2}\left\langle\varphi(t_1,x_1)\varphi(t_2,x_2)\cdots\varphi(t_n,x_n)\right\rangle$$

to be UV finite. But, this is sufficient also for equal-point product,

$$Z_\varphi^{n/2}\left\langle\varphi(t_1,x_1)\varphi(t_2,x_1)\cdots\varphi(t_n,x_n)\right\rangle.$$

- We can actually identify this GF and ERG.

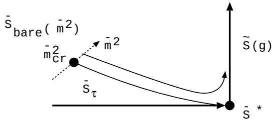
- The evolution of the Wilson action $S_{\tau=0} \to S_\tau$ in dimensionless variables can be formulated as the equality ($\eta_\tau = d \ln Z_\tau / d\tau$):

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\tau}^{K,k} = e^{-\tau n(D+2)/2} Z_\tau^{n/2} \left\langle\left\langle \phi(e^{-\tau} p_1) \cdots \phi(e^{-\tau} p_n) \right\rangle\right\rangle_{S_{\tau=0}}^{K,k},$$

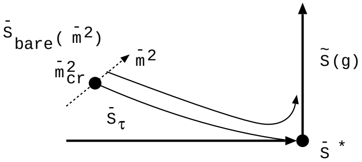
where the modified correlation function is defined by

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_S^{K,k} \equiv \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \exp\left[ -\int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \phi(p_1) \cdots \phi(p_n) \right\rangle_S .$$

- We now construct a continuum theory around a fixed point (e.g., the Wilson–Fisher fixed point) $S^*[\phi]$.



# Continuum limit around the fixed point $S^*$




- We assume that the critical surface associated with the fixed point $S^*$ is approached by

$$m^2 \to m_{\text{cr}}^2, \qquad \xi = c(m^2 - m_{\text{cr}}^2)^{-1/y},$$

i.e.,

$$S_{\tau=0}(m_{\text{cr}}^2) \stackrel{\tau\to\infty}{\longrightarrow} S^*.$$

- The renormalized trajectory of the Wilson action, $\tilde{S}(g)$ specified by a relevant parameter $g$, is then defined by a particular limit,

$$\tilde{S}(g) \equiv \lim_{\tau\to\infty} S_\tau(m^2(g, e^\tau)), \qquad m^2(g, e^\tau) \equiv m_{\text{cr}}^2 + ge^{-y\tau}, \qquad \xi \sim e^\tau.$$

# The equality in continuum limit

- Let us now rewrite the equality

$$\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle^{K,k}_{S_\tau} = e^{-\tau n(D+2)/2} Z_\tau^{n/2} \langle\!\langle \phi(p_1 e^{-\tau})\cdots\phi(p_n e^{-\tau})\rangle\!\rangle^{K,k}_{S_{\tau=0}}$$

in terms of dimensionful variables. For RHS, we introduce a "bare" continuum action by

$$S_{\Lambda_0}[\phi] \equiv S_{\tau=0}(m^2(g,\Lambda_0/\mu))[\phi(p)\to\Lambda_0^{(D+2)/2}\phi(\Lambda_0 p)],$$

and for LHS, a Wilson action with finite cutoff $\Lambda$,

$$\begin{aligned} S_\Lambda[\phi] &= \lim_{\Lambda_0\to\infty} S_{\tau=\ln(\Lambda_0/\Lambda)}(m^2(g,\Lambda_0/\mu))[\phi(p)\to\Lambda^{(D+2)/2}\phi(\Lambda p)] \\ &= \lim_{\Lambda_0\to\infty} S_{\ln(\Lambda_0/\Lambda)}(m^2(g_\Lambda,\Lambda_0/\Lambda))[\phi(p)\to\Lambda^{(D+2)/2}\phi(\Lambda p)], \end{aligned}$$

where

$$g_\Lambda \equiv g\left(\frac{\mu}{\Lambda}\right)^y,$$

specifies a point on the renormalized trajectory.

# The equality in continuum limit

- The equality of the modified correlation function then becomes (here, we assume $\eta_\tau = \eta =$ const. for simplicity)

$$\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_\Lambda}^{K_\Lambda,k_\Lambda} = \lim_{\Lambda_0\to\infty}\left(\frac{\Lambda_0}{\Lambda}\right)^{n\eta/2}\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_{\Lambda_0}}^{K_{\Lambda_0},k_{\Lambda_0}},$$

where $K_\Lambda(p) \equiv K(p/\Lambda)$ and $k_\Lambda(p) \equiv k(p/\Lambda)$ (and $\Lambda \to \Lambda_0$, resp.)

- This reads, in terms of the ordinary correlation function,

$$\left\langle \exp\left\{-\int_p \frac{k(p/\Lambda)}{p^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right\}\phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda}$$

$$= \lim_{\Lambda_0\to\infty}\left(\frac{\Lambda_0}{\Lambda}\right)^{n\eta/2}\prod_{i=1}^{n}\underbrace{K(p_i/\Lambda)}_{\to e^{-tp_i^2}}\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_{\Lambda_0}}$$

- Finally, taking the cutoff function

$$K(p) = e^{-p^2},$$

and identifying the GF time and the cutoff by

$$t \equiv \frac{1}{\Lambda^2},$$

# Diffused scalar field $\varphi$ emerges

- ..., we have

$$\lim_{\Lambda_0\to\infty} \underbrace{\left(\frac{\Lambda_0}{\Lambda}\right)^{\eta}}_{Z_\varphi} \left\langle \underbrace{e^{-tp^2}\phi(p)}_{\varphi(t,p)} \underbrace{e^{-tq^2}\phi(q)}_{\varphi(t,q)} \right\rangle_{S_{\Lambda_0}} = \langle\phi(p)\phi(q)\rangle_{S_\Lambda} - \delta(p+q)\frac{k(p/\Lambda)}{p^2},$$

$$\lim_{\Lambda_0\to\infty} \underbrace{\left(\frac{\Lambda_0}{\Lambda}\right)^{n\eta/2}}_{{Z_\varphi}^{n/2}} \left\langle \underbrace{e^{-tp_1^2}\phi(p_1)}_{\varphi(t,p_1)} \cdots \underbrace{e^{-tp_n^2}\phi(p_n)}_{\varphi(t,p_n)} \right\rangle_{S_{\Lambda_0}}^{\text{conn}} = \langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}^{\text{conn}}.$$

- Correlation functions of the diffused field $\varphi$ w.r.t. the bare action $S_{\Lambda_0}$ are equal to those of bare fields w.r.t. the Wilson action $S_\Lambda$, up to the "contact term" in the 2-point function.
- LHS is finite, because in RHS $S_\Lambda$ is a Wilson action with finite cutoff $\Lambda$.
- The diffused scalar field actually requires the wave function renormalization $Z_\varphi$.
- This finiteness persists even for equal-point product because $S_\Lambda$ is a Wilson action with finite cutoff $\Lambda$.
- Even for a nontrivial (i.e., not Gaussian) fixed point...
- "GF" $\Leftrightarrow$ ERG (i.e., the Polchinski equation) in scalar field theory

# Generalization to gauge theory?

- ... seems difficult because

$$\frac{\partial}{\partial t}B_\mu^a(t,x) = D_\nu G_{\nu\mu}^a(t,x)$$

  is highly nonlinear.
- The above connection of GF and ERG in scalar field theory nevertheless provides an interesting representation of $S_\tau$:

$$e^{S_\tau[\phi]} = \exp\left[\int d^Dx \int d^Dy\ \mathcal{D}(x-y)\frac{1}{2}\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\right]$$
$$\times \int [d\phi'] \prod_{x'} \delta\left(\phi(x) - e^{\tau(D-2)/2}Z_\tau^{1/2}\varphi'(t,x'e^\tau)\right)$$
$$\times \exp\left[-\int d^Dx'' \int d^Dy''\ \mathcal{D}(x''-y'')\frac{1}{2}\frac{\delta^2}{\delta\phi'(x'')\delta\phi'(y'')}\right] e^{S_{\tau=0}[\phi']},$$

  where

$$\mathcal{D}(x) \equiv \int_p e^{ipx}\frac{k(p)}{p^2},$$

  and

$$\partial_t\varphi'(t,x) = \partial^2\varphi'(t,x), \qquad \varphi'(t=0,x) = \phi'(x), \qquad t \equiv e^{2\tau} - 1.$$
- Wilson action $S_\tau[\phi]$ is directly expressed by the diffused field $\varphi$.

# Gradient flow exact renormalization group (GFERG) for Yang–Mills theory

- We may imitate the above representation as

$$e^{S_\tau[A]} = \exp\left[\int d^Dx\,\frac{1}{2}\frac{\delta^2}{\delta A^a_\mu(x)\delta A^a_\mu(x)}\right]$$
$$\times\int[dA']\prod_{x',\nu,b}\delta\left(A^b_\nu(x') - e^{\tau(D-2)/2}z_\tau^{1/2}B'^b_\nu(t,x'e^\tau)\right)$$
$$\times\exp\left[-\int d^Dx''\,\frac{1}{2}\frac{\delta^2}{\delta A'^c_\rho(x'')\delta A'^c_\rho(x'')}\right]e^{S_{\tau=0}[A']},$$

  where

$$\frac{\partial}{\partial t}B'^a_\mu(t,x) = D_\nu G'^a_{\nu\mu}(t,x)+\alpha_0 D_\mu\partial_\nu B'^a_\nu(t,x),\quad B'^a_\mu(t=0,x) = A'^a_\mu(x),\quad t\equiv e^{2\tau}-1,$$

  where $\alpha_0$ is a “gauge fixing parameter” (see below).

- The above corresponds to the choice

$$k(p) = p^2.$$

- This defines GFERG (see below).

# Properties of GFERG

- Partition function is preserved under the evolution (thus the RG flow is "exact"):

$$Z = \int [dA]\, e^{S_\tau[A]} = \int [dA]\, e^{S_{\tau=0}[A]}.$$

- Gauge invariance: $S_\tau[A]$ is invariant under

$$A_\mu^a(x) \to A_\mu^a(x) + \lambda_\tau^{-1} z_\tau^{1/2} \partial_\mu \omega^a(x) + f^{abc} A_\mu^b(x) \omega^c(x),$$

  where

$$\lambda_\tau \equiv e^{-\tau(D-4)/2},$$

  if $S_{\tau=0}[A]$ is invariant under

$$A_\mu^a(x) \to A_\mu^a(x) + \partial_\mu \omega^a(x) + f^{abc} A_\mu^b(x) \omega^c(x).$$

- $S_\tau[A]$ is independent of the parameter $\alpha_0$, if $S_{\tau=0}[A]$ is gauge invariant in the above sense.
- From the analogue to the scalar field theory and the finiteness of GF, we expect that correlation functions in the continuum limit are all finite (possibly with an appropriate choice of $z_\tau$); but has no explicit proof yet.

# Gauge invariance of $S_\tau[A]$

- First,
$$\exp\left[\int d^Dx\,\frac{1}{2}\frac{\delta^2}{\delta A_\mu^a(x)\delta A_\mu^a(x)}\right]$$
is invariant, because $\delta/\delta A_\mu^a(x)$ transforms in the adjoint representation.
- Next, the argument of the delta function transforms as
$$\begin{aligned}
&A_\nu^b(x) - e^{\tau(D-2)/2} z_\tau^{1/2} B_\nu^{\prime b}(t, xe^\tau)\\
&\to A_\nu^b(x) + \lambda_\tau^{-1} z_\tau^{1/2}\partial_\nu\omega^b(x) + f^{bcd}A_\nu^c(x)\omega^d(x) - e^{\tau(D-2)/2} z_\tau^{1/2} B_\nu^{\prime b}(t, xe^\tau)\\
&= A_\nu^b(x) - e^{\tau(D-2)/2} z_\tau^{1/2}\left[B_\nu^{\prime b}(t, xe^\tau) - e^{-\tau}\partial_\nu\omega^b(x) - e^{-\tau(D-2)/2} z_\tau^{-1/2} f^{bcd} A_\nu^c(x)\omega^d(x\right.\\
&= A_\nu^b(x) - e^{\tau(D-2)/2} z_\tau^{1/2}\left[B_\nu^{\prime b}(t, xe^\tau) - \frac{\partial}{\partial(x_\nu e^\tau)}\omega^b(x) - f^{bcd} B_\nu^{\prime c}(t, xe^\tau)\omega^d(x)\right].
\end{aligned}$$
- This induces the gauge transformation on $B'$:
$$B_\mu^{\prime a}(t,x) \to B_\mu^{\prime a}(t,x) - D'_\mu\omega^a(xe^{-\tau}).$$
- This then induces the gauge transformation on the initial value $A'$ Through the GF equation:
$$A_\mu^{\prime a}(x) \to A_\mu^{\prime a}(x) - D'_\mu\xi^a(0,x),\quad (\partial_s - \alpha_0 D_\nu\partial_\nu)\xi^a(s,x) = 0,\quad \xi^a(s=t,x) = \omega^a(xe^{-\tau}).$$
Therefore, if $S_{\tau=0}[A]$ is gauge invariant, then the Wilson action $S_\tau[A]$ is gauge invariant.

- Taking the derivative of $S_\tau[A]$ w.r.t. $\tau$, we have

$$\frac{\partial}{\partial\tau} e^{S_\tau[A]}$$
$$= \int d^D x \, \frac{\delta}{\lambda_\tau z_\tau^{-1/2} \delta A_\mu^a(x)}$$
$$\times \left[ -2D_\nu F_{\nu\mu}^a(x) - 2\alpha_0 D_\mu \partial_\nu A_\nu^a(x) \right.$$
$$\left. - \left( \frac{D-2}{2} + \frac{\zeta_\tau}{2} + x \cdot \frac{\partial}{\partial x} \right) A_\mu^a(x) \right] \Bigg|_{A \to \lambda_\tau z_\tau^{-1/2}(A + \delta/\delta A)} e^{S_\tau[A]},$$

where

$$\zeta_\eta \equiv \frac{d}{d\tau} \ln z_\tau .$$

- This equation does not refer to the initial action $S_{\tau=0}[A]$ as should be an ERG equation.
- Containing the functional derivative to 4th order.
- This ERG equation in Yang–Mills theory preserves the gauge invariance as we have shown.

# Perturbative solution to GFERG

- The parameter
$$\lambda_\tau = e^{-\tau(D-4)/2},$$
can be used as an expansion parameter for $S_\tau[A]$ as
$$S_\tau[A] \equiv \sum_{n=2}^{\infty} \lambda_\tau^{n-2} \frac{1}{n!} \int d^D x_1 \cdots \int d^D x_n \, C_{n,\mu_1\cdots\mu_n}^{a_1\cdots a_n}(\tau; x_1, \dots, x_n) A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n),$$
where $w_n = O(\lambda_\tau^0)$.
- It is important to first understand this solution, before attacking more difficult nonperturbative solutions.
- The $O(\lambda_\tau^0)$ term of $C_2$:
$$C_{2,\mu\nu}^{ab}(\tau; x, y) = -\delta^{ab} \int_p e^{ip(x-y)} \left[ \frac{1}{C(pe^{-\tau})e^{-2p^2} + p^2} \overbrace{(p^2\delta_{\mu\nu} - p_\mu p_\nu)}^{\text{gauge inv. part}} + \frac{1}{D(pe^{-\tau})e^{-2\alpha_0 p^2} + p^2} p_\mu p_\nu \right],$$
where $C(p)$ and $D(p)$ are arbitrary functions of $p^2$. Locality demands that
$$C(p) = C_0 + C_1 p^2 + \frac{1}{2} C_2 (p^2)^2 + \cdots, \quad D(p) = D_0 + D_1 p^2 + \frac{1}{2} D_2 (p^2)^2 + \cdots.$$
- $\tau \to \infty$ approaches the Gaussian fixed point; $C_0$ and $D_0$ are marginal parameters.

# Some lower-order terms (Sonoda, H.S., work in progress)

- $O(\lambda_\tau^1)$ term of $C_3$ is already very complicated; a gauge-invariant local solution (with an appropriate initial condition) is

$$
\begin{aligned}
&C^{abc}_{3,\mu\nu\rho}(\tau;p_1,p_2,p_3)\\
&= if^{abc}\frac{e^{-p_1^2}}{e^{-2p_1^2}C_0+p_1^2}\frac{e^{-p_2^2}}{e^{-2p_2^2}C_0+p_2^2}\frac{e^{-p_3^2}}{e^{-2p_3^2}C_0+p_3^2}\\
&\quad\times\Biggl(C_0^2\left[1+p_1^2F(p_1;p_2,p_3)+p_2^2F(p_2;p_3,p_1)+p_3^2F(p_3;p_1,p_2)\right]\\
&\qquad\qquad\times[\delta_{\mu\nu}(p_1-p_2)_\rho+\delta_{\nu\rho}(p_2-p_3)_\mu+\delta_{\rho\mu}(p_3-p_1)_\nu]\\
&\qquad+C_0\Bigl\{\left[F(p_1;p_2,p_3)C_0+F(p_3;p_1,p_2)C_0-e^{p_1^2-p_2^2+p_3^2}\right](p_1^2\delta_{\mu\nu}-p_{1\mu}p_{1\nu})p_{3\rho}\\
&\qquad\qquad-\left[F(p_2;p_3,p_1)C_0+F(p_3;p_1,p_2)C_0-e^{-p_1^2+p_2^2+p_3^2}\right](p_2^2\delta_{\mu\nu}-p_{2\mu}p_{2\nu})p_{3\rho}\\
&\qquad\qquad\qquad+\text{cyclic in 1,2,3 and }\mu,\,\nu,\,\rho\Bigr\}\\
&\qquad-\frac{1}{2}e^{p_1^2+p_2^2+p_3^2}\left\{\left[p_1^2(p_2-p_3)_\mu-p_{1\mu}p_1\cdot(p_2-p_3)\right]p_{2\nu}p_{3\rho}+\text{cyclic in 1,2,3 and }\mu,\,\nu,\,\rho\right\}\Biggr),
\end{aligned}
$$

where

$$
F(p_1;p_2,p_3)\equiv\frac{e^{p_1^2-p_2^2-p_3^2}-1}{p_1^2-p_2^2-p_3^2}.
$$

- We also have obtained $O(\lambda_\tau^2)$ term of $C_4$.

- The idea is basically the same:

$$
\begin{aligned}
e^{S_\tau[A,\psi,\bar\psi]} &= \exp\left[\int d^Dx\,\frac{1}{2}\frac{\delta^2}{\delta A^a_\mu(x)\delta A^a_\mu(x)}\right]\exp\left[-i\int d^Dx'\,\frac{\delta}{\delta\psi(x')}\,\not{D}_\tau\,\frac{\delta}{\delta\bar\psi(x')}\right]\\
&\quad\times\int[dA'd\psi'd\bar\psi']\prod_{x'',\nu,b}\delta\left(A^b_\nu(x'')-e^{\tau(D-2)/2}z_\tau^{1/2}B'^b_\nu(t,x''e^\tau)\right)\\
&\qquad\times\delta\left(\psi(x'')-e^{\tau(D-1)/2}Z_\tau^{1/2}\chi'(t,x''e^\tau)\right)\\
&\qquad\times\delta\left(\bar\psi(x'')-e^{\tau(D-1)/2}Z_\tau^{1/2}\bar\chi'(t,x''e^\tau)\right)\\
&\qquad\times\exp\left[i\int d^Dx'''\,\frac{\delta}{\delta\psi'(x''')}\,\not{D}_{\tau=0}\,\frac{\delta}{\delta\bar\psi'(x''')}\right]\\
&\qquad\quad\times\exp\left[-\int d^Dx''''\,\frac{1}{2}\frac{\delta^2}{\delta A'^c_\rho(x'''')\delta A'^c_\rho(x'''')}\right]e^{S_{\tau=0}[A',\psi',\bar\psi']}.
\end{aligned}
$$

- The diffused fermion fields, $\chi'$ and $\bar\chi'$, are given by the fermion flow from $\psi'$ and $\bar\psi'$.
- Gauge and chiral symmetries are preserved; but how chiral anomaly emerges?

# Summary

- We clarified a connection between GF and ERG in scalar field theory.
- This connection provides an intuitive understanding on the finiteness of GF.
- Imitating the structure of this connection, we proposed GFERG in pure Yang–Mills theory.
- GFERG preserves manifest gauge invariance.
- GFERG can be generalized to lattice gauge theory (see our paper).
- Fermion (and matter fields) can be incorporated.

# Many many things to be investigated; as they occur to my mind...

- $\beta$ function in at least 1-loop order should be reproduced (ongoing).
- Want to address finiteness of GFERG in gauge theory from finiteness of GF (at the Gaussian fixed point).
- Wetterich equation for the average effective action is “more popular” than Polchinski’s. Is it possible to construct GF version of the Wetterich equation?
- With GFERG, a gauge-invariant truncation of the Wilson action should be possible. Can we address nontrivial fixed points in gauge theory? in QED?
- Small flow-time representation of EMT at nontrivial fixed points?
- Generalization to gravity based on the Ricci flow? Reparametrization-invariant critical exponents?