Stochastic computation of g-2 in QED

Ryuichiro Kitano (KEK)
based on 2103.10106 with H.Takaura (KEK) and S.Hashimoto (KEK)

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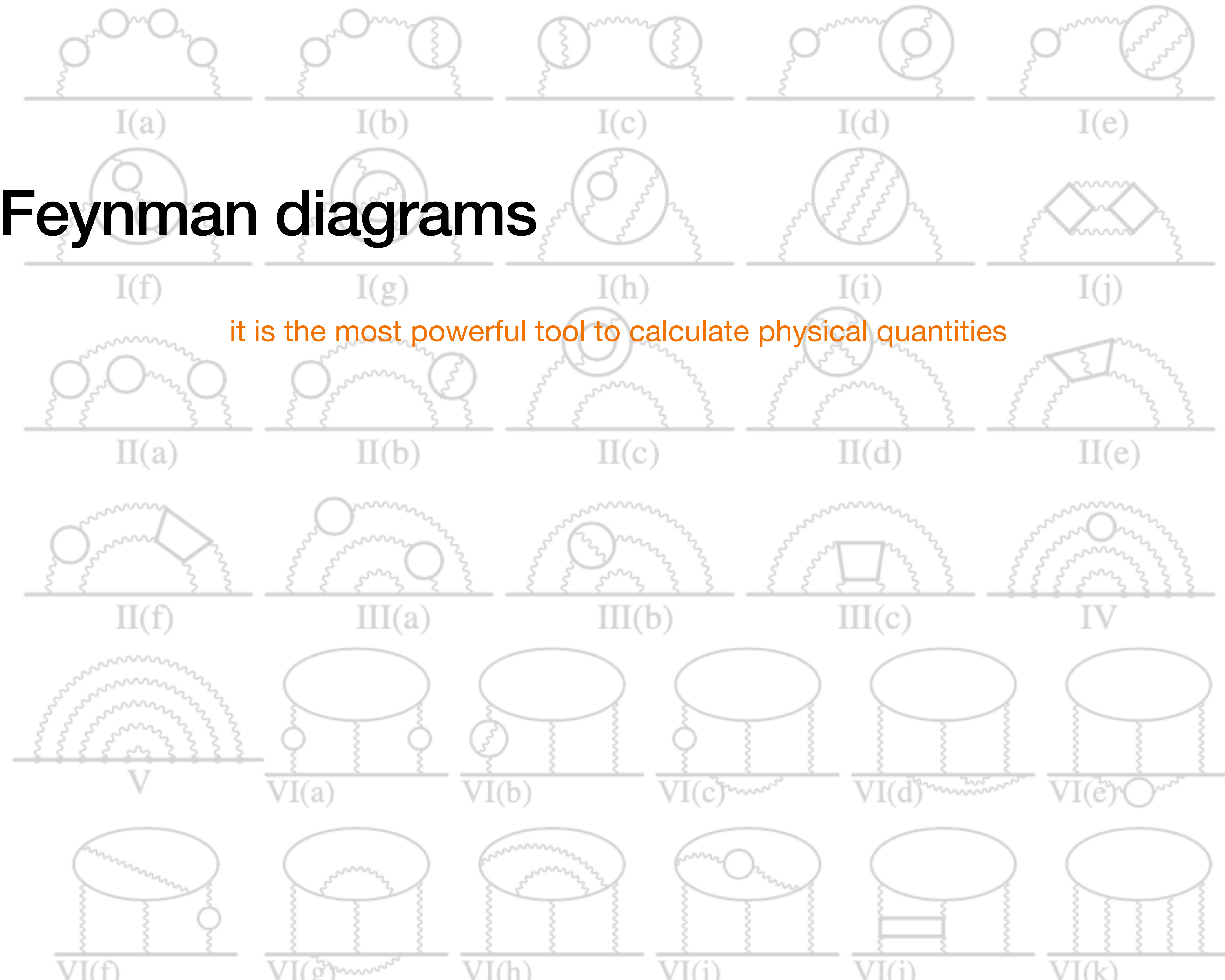
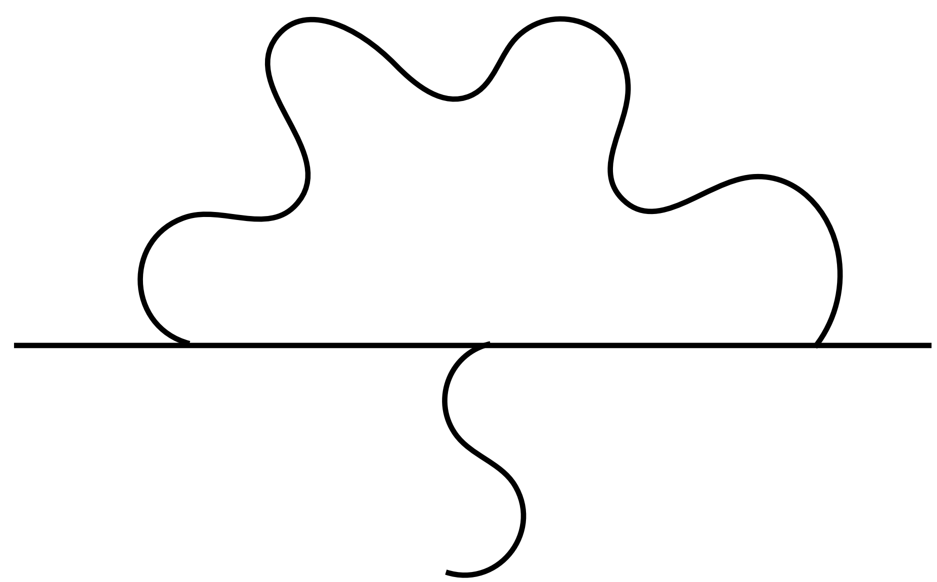
Taken from Aoyama, Kinoshita and Nio '19

Feynman diagrams

it is the most powerful tool to calculate physical quantities



Draw diagrams,
multiply propagators and vertices, and
integrate over loop momenta.



In QED, this diagram gives O(α) correction to the Dirac's g=2.

$$g - 2 = \frac{\alpha}{\pi}$$

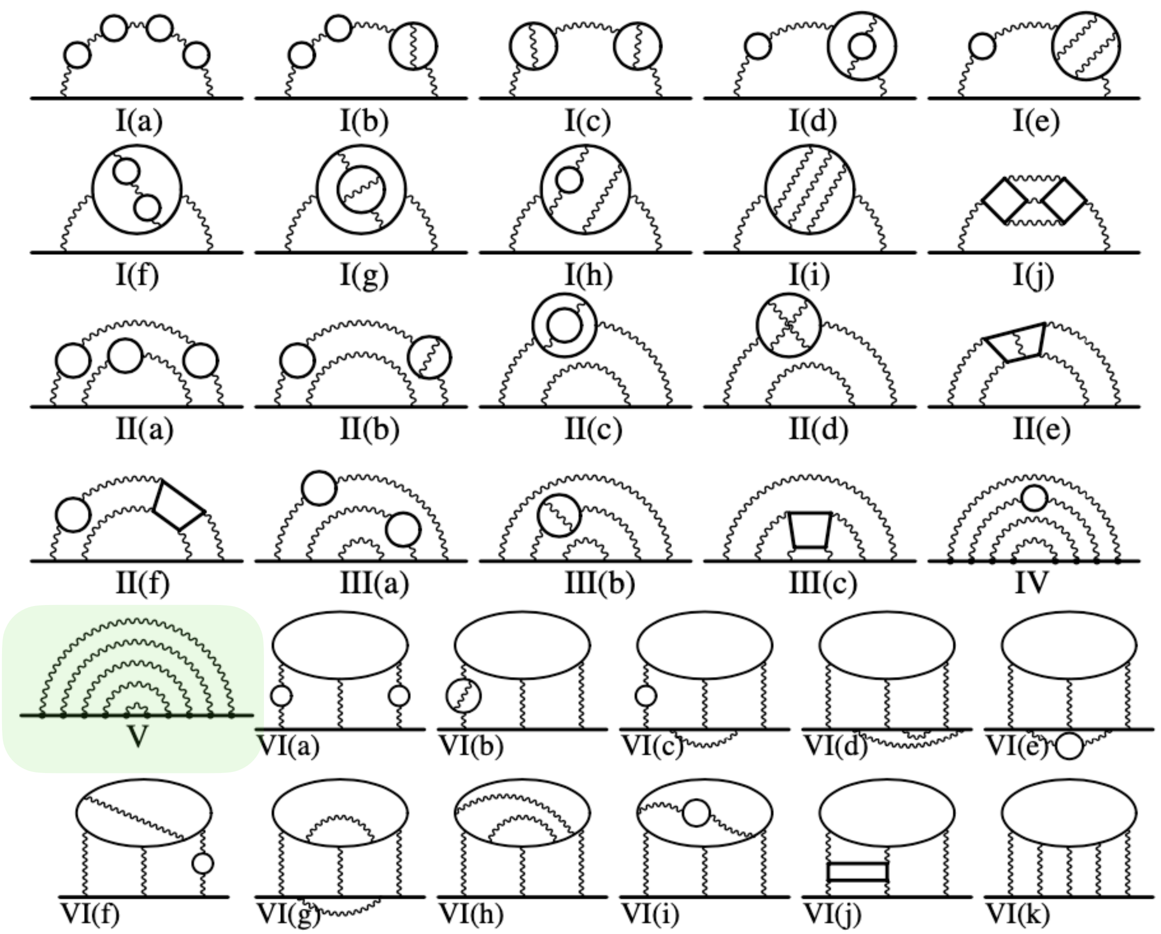
(Homework)

[Schwinger '48]

[Aoyama, Hayakawa, Kinoshita, Nio..]
[taken from Review: Aoyama, Kinoshita, Nio '19]

State-of-the-art computation:

5-loops **12,672** diagrams!



A = (g-2)/2

$$A_i = \left(\frac{\alpha}{\pi}\right) A_i^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A_i^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A_i^{(6)} + \cdots, \qquad \text{for } i = 1,2,3\,.$$

1-loop 2-loop 3-loop ....

Coefficient $A_i^{(2n)}$	Value (Error)	References	
$A_1^{(2)}$	0.5	[5]	1 diagram
$A_2^{(2)}(m_e/m_\mu)$	0		
$A_2^{(2)}(m_e/m_\tau)$	0		
$A_3^{(2)}(m_e/m_\mu, m_e/m_\tau)$	0		
$A_1^{(4)}$	−0.328 478 965 579 193 ···	[23,24]	7 diagrams
$A_2^{(4)}(m_e/m_\mu)$	0.519 738 676 (24) × 10^{-6}	[27]	
$A_2^{(4)}(m_e/m_\tau)$	0.183 790 (25) × 10^{-8}	[27]	
$A_3^{(4)}(m_e/m_\mu, m_e/m_\tau)$	0		
$A_1^{(6)}$	1.181 241 456 587 ···	[25,33]	72 diagrams
$A_2^{(6)}(m_e/m_\mu)$	−0.737 394 164 (24) × 10^{-5}	[28–31]	
$A_2^{(6)}(m_e/m_\tau)$	−0.658 273 (79) × 10^{-7}	[28–31]	
$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$	0.1909 (1) × 10^{-12}	[43]	
$A_1^{(8)}$	−1.912 245 764 ···	[26,39]	891 diagrams
$A_2^{(8)}(m_e/m_\mu)$	0.916 197 070 (37) × 10^{-3}	[32,35]	
$A_2^{(8)}(m_e/m_\tau)$	0.742 92 (12) × 10^{-5}	[32,35]	
$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$	0.746 87 (28) × 10^{-6}	[32,35]	
$A_1^{(10)}$	6.737 (159)	new,[40]	
$A_2^{(10)}(m_e/m_\mu)$	−0.003 82 (39)	[35,39]	
$A_2^{(10)}(m_e/m_\tau)$	$\mathcal{O}(10^{-5})$		12,672 diagrams
$A_3^{(10)}(m_e/m_\mu, m_e/m_\tau)$	$\mathcal{O}(10^{-5})$		

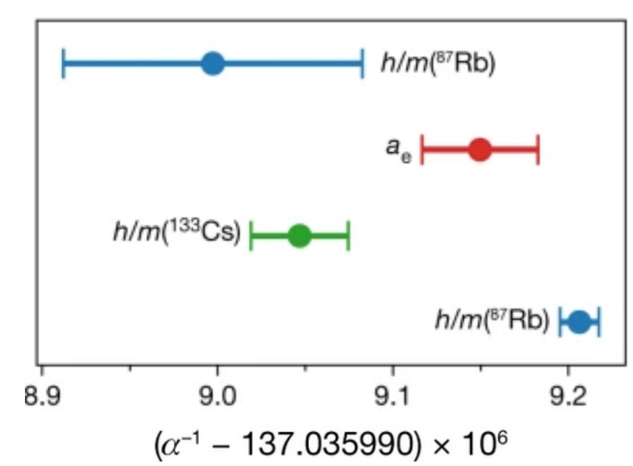
6-loop 202,770 diagrams

electron g-2:

agreed!

$$a_e(\text{theory} : \alpha(\text{Rb})) = 1\,159\,652\,182.037\,(720)(11)(12) \times 10^{-12},$$
$$a_e(\text{theory} : \alpha(\text{Cs})) = 1\,159\,652\,181.606\,(229)(11)(12) \times 10^{-12},$$
$$a_e(\text{expt.}) = 1\,159\,652\,180.73\,(28) \times 10^{-12}.$$ [Harvard '08]

1 2 3 4 5-loop



[Recent Rb update, Morel et al, '20]

There seems to be a discrepancy in 5-loop computations between two groups…

[Volkov '19]

both Monte Carlo integration and analytical calculations. For example, the uncertainty in (2) is entirely determined by that contribution. Also, it is the contribution that suffered the most from found mistakes and corrections; see Ref. [34]. The value

$$A_1^{(10)}[\text{no lepton loops, AKN}] = 7.668(159). \tag{3}$$

can be obtained by using (2) and the value of the remaining part that can be extracted from Ref. [34]. By 2019, there was no independent calculations of $A_1^{(10)}[\text{no lepton loops}]$.

We recalculated this contribution with the help of the supercomputer "Govorun" (JINR, Dubna, Russia). 40000 GPU-hours of Monte Carlo integration on NVidia Tesla V100 that were spread over several months have led to the result

$$A_1^{(10)}[\text{no lepton loops, Volkov}] = 6.793(90), \tag{4}$$

where the uncertainty corresponds to 1σ limits. It is in good agreement with the preliminary value 6.782(113) published in Ref. [35]. The descrepancy between this result and (3) is approximately 4.8σ. This means that the values are probably different. The reason of this difference is unknown. Sec.

Nice to have some **independent** methods to calculate this?

I'll try to develop a numerical method to evaluate the perturbative coefficients in QED, **which does not use the Feynman diagrams** (because I'm lazy)**.**

What I'm going to explain today is really just QED computations. Nothing fancy or deep. But, **it's fun!**

It is going to be a lattice calculation. But don't sleep now. It won't be too technical.

[Di Renzo, Marchesini, Marenzoni, Onofri ’94, Di Renzo, Scorzato ’00, …]

Stochastic Numerical Perturbation Theory

Path integral quantization:

$$\langle \phi(x)\phi(y)\ldots\rangle = \int [d\phi]\phi(x)\phi(y)\ldots e^{-S[\phi]} \Big/ \int [d\phi] e^{-S[\phi]}$$

Equivalent! [Parisi, Wu ’81]

Stochastic quantization:

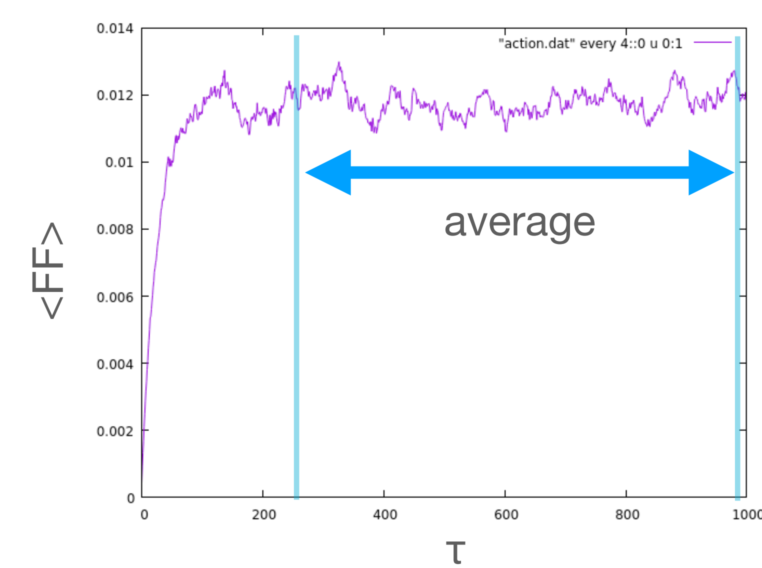
$$\langle \phi(x)\phi(y)\ldots\rangle = \langle \phi(x,\tau)\phi(y,\tau)\ldots\rangle_{\text{stochastic}}$$

Long “time” average

$$\frac{\partial \phi(x,\tau)}{\partial \tau} = -\frac{\delta S}{\delta \phi(x,\tau)} + \eta(x,\tau)$$

(Langevin equation)

“time” evolution — Eq. of motion — Gaussian noise



Basically, the field value scans around the **classical solutions.**

Averaging them gives **quantum** correlators!

Now perturbative expansion:

coupling constant

$$\langle \phi(x)\phi(y)\ldots\rangle = \langle \phi(x)\phi(y)\ldots\rangle^{(0)} + \lambda\langle \phi(x)\phi(y)\ldots\rangle^{(1)} + \cdots$$

We want to calculate these.

Recipe:

Expand fields by couplings

$$\phi(x,\tau) = \phi^{(0)}(x,\tau) + \lambda\phi^{(1)}(x,\tau) + \cdots$$

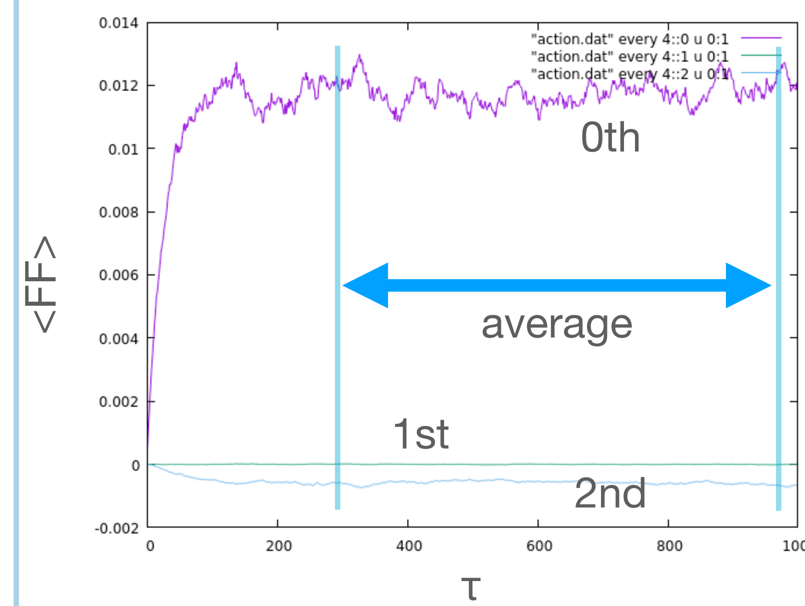
Solve the Langevin equations for **each $\phi^{(n)}$** numerically

Random noise (only for the lowest order)

$$\frac{\partial\phi^{(n)}(x,\tau)}{\partial\tau} = -\frac{\delta S}{\delta\phi(x)}\bigg|^{(n)}_{\phi=\phi(x,\tau)} + \eta(x,\tau)\delta_{n,0}$$

Combine

$$\langle \phi(x)\phi(y)\ldots\rangle^{(n)} = \sum_{n_1+n_2+\cdots=n} \langle \phi^{(n_1)}(x)\phi^{(n_2)}(y)\ldots\rangle_{\text{stochastic}}$$



Yes, that's it!

What we need to do is to solve a set of **stochastic differential equations numerically.** Very simple.

The degrees of freedom can be made finite by putting the theory on the **lattice**.

Indeed, in lattice QCD the expectation value of the Wilson loop has been calculated up to $O(\alpha_S^{35})$!

[Bali, Bauer, Pineda, '14, Del Debbio, Renzo, Filaci '18]

Maybe even simpler in QED?

Anyway, let's try.

Lattice unit:

Our Lattice QED action: $S_{\text{lattice}} = S_{\text{g}} + S_{\text{gf}} + S_{\text{mass}} + S_{\text{f}},$ $(a = 1)$

UV regulator (suppress log divergence)

$$S_{\text{g}} = \frac{1}{4} \sum_{n,\mu,\nu} \left[e^{-\nabla^2/\Lambda_{\text{UV}}^2} (\nabla_\mu A_\nu(n) - \nabla_\nu A_\mu(n)) \right]^2,$$

kinetic term gets large for high virtuality $k^2 >> \Lambda_{UV}^2$

Non-compact QED action

(Link variable U not used here. Simple.)

$$\nabla_\mu f(n) = f(n+\hat{\mu}) - f(n), \quad \nabla_\mu^* f(n) = f(n) - f(n-\hat{\mu}),$$

$$S_{\text{gf}} = \frac{1}{2\xi} \sum_n \left[e^{-\nabla^2/\Lambda_{\text{UV}}^2} \sum_\mu \nabla_\mu^* A_\mu(n) \right]^2,$$

Gauge fixing term

(Necessary. Otherwise A_μ random walks in the gauge direction.)

$$S_{\text{mass}} = \frac{1}{2} \sum_{n,\mu} m_\gamma^2 \left[e^{-\nabla^2/\Lambda_{\text{UV}}^2} A_\mu(n) \right]^2,$$

Photon mass (IR regulator)

(Necessary. Otherwise zero mode part of A_μ random walks. Gauge invariance broken, but in a rather controlled way.)

This factor takes care of **doublers**. Justified in perturbative calculations.

$$S_{\text{f}} = -\frac{1}{16} \ln \det D.$$

Fermion determinant

(Fermions loops are all included)

Dirac operator:

$$(D)_{nm} = m\delta_{nm} + \frac{1}{2} \sum_\mu \left[\gamma_\mu e^{ieA_\mu(n)} \delta_{n+\hat{\mu},m} - \gamma_\mu e^{-ieA_\mu(n-\hat{\mu})} \delta_{n-\hat{\mu},m} \right],$$

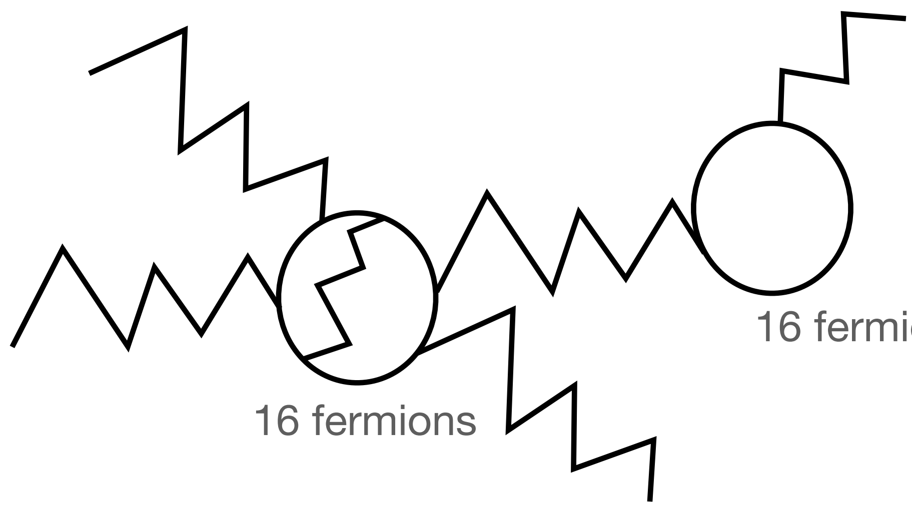
Link variable, U, is used here for gauge invariance.

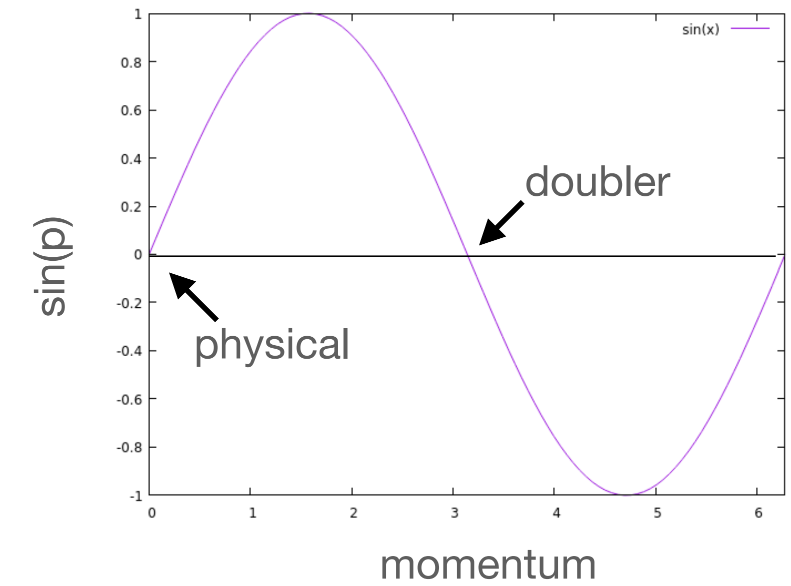
Naive fermion

(There are **16 doublers**. **Chiral symmetry** is maintained on the lattice.)

Doublers?

Momentum space propagator: $\dfrac{-i\sin(p_\mu)\gamma_\mu + m}{\sin^2(p_\mu) + m^2}$ $2^4 = 16$ poles!





16 fermions run in the loops. → x **1/16** for each loop gives the correct answer.

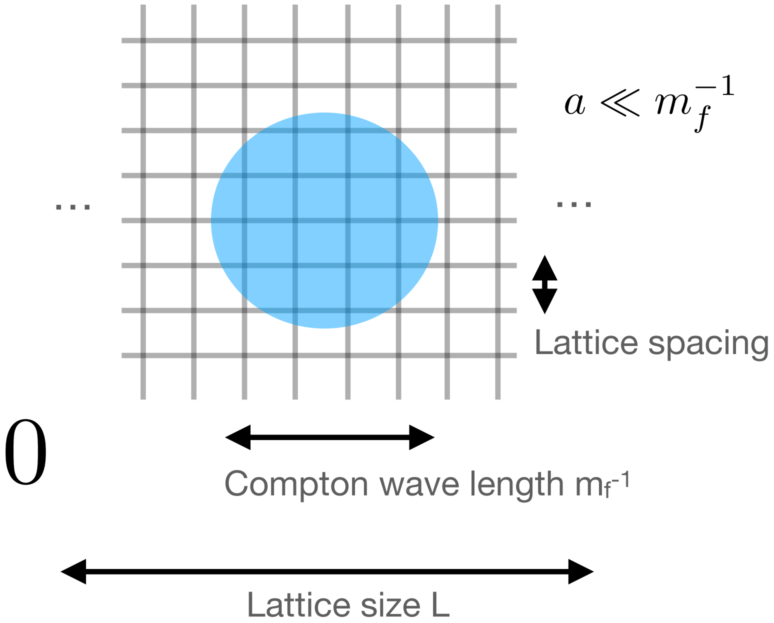
$$-\ln\det D \to -\frac{1}{16}\ln\det D$$

Wow. That sounds a pretty rude idea.

But it is **just true in the perturbative calculations** as long as γ_5 is not involved.

→ OK in QED

Continuum limit:



Physical quantity

Energy scale in question: m_f (Fermion mass)

One should keep this finite while the lattice spacing to be zero. $m_f a \to 0$

Chiral symmetry is maintained on the lattice: $m_f \propto m$

Parameter in the Lagrangian
No additive renormalization

Continuum limit is realized by taking **small ma** in the Lagrangian.

$$ma \to 0$$

Note:
Coupling constant “e” is not a parameter.
We formally expand everything in terms of “e”. There is nowhere we need an explicit value of “e”.

Also, one should take the limits of

Zero photon mass: $m_\gamma / m \to 0$

Large volume: $L \to \infty \quad T \to \infty$

Infinite UV cutoff: $\Lambda_{\mathrm{UV}} / m \to \infty$

We’ll come back how those limits are taken.

Langevin equation

Perturbative expansion:

$$A_\mu(n,\tau) = \sum_{p=0}^{\infty} e^p A_\mu^{(p)}(n,\tau),$$

Discretize the τ direction.

We update the gauge field A_μ according to the equation:

For example,

$$\left.\frac{\delta S_{\rm g}}{\delta A_\mu(n)}\right|_{(p)} = e^{-2\nabla^2/\Lambda_{\rm UV}^2} \sum_\nu \left(\nabla_\nu^* \nabla_\mu A_\nu^{(p)}(n) - \nabla_\nu^* \nabla_\nu A_\mu^{(p)}(n)\right)$$

$$\frac{\partial A_\mu(n,\tau)}{\partial \tau} = -\frac{\delta S_{\rm lattice}}{\delta A_\mu(n,\tau)} + \eta_\mu(n,\tau).$$

Noise

Pretty simple except for

$$\frac{\delta S_{\rm f}}{\delta A_\mu(n)} = -\frac{1}{16}{\rm Tr}\left(\frac{\delta D}{\delta A_\mu(n)} D^{-1}\right).$$

We need to calculate the **inverse of the Dirac operator** for each Langevin step.

That can be quite effectively done by using the following **recursion formula** and **FFT**: [Di Renzo, Scorzato '00]

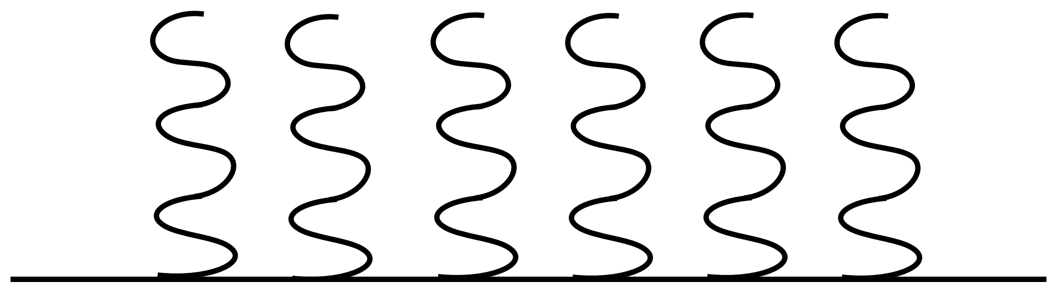
$$\left.(D^{-1})\right|_{(0)} = D_0^{-1}, \quad \left.(D^{-1})\right|_{(p)} = -\left.\left[D_0^{-1} D \sum_{q=0}^{p-1} e^q \left.(D^{-1})\right|_{(q)}\right]\right|_{(p)}.$$

Well, it's essentially the same as the usual perturbative expansion:

[Di Renzo, Scorzato '00]

D^{-1} : propagator

$\sim \frac{1}{i\partial\!\!\!/ + m + ieA\!\!\!/}$



diagonal in the position space.

FFT ($x \leftarrow p$) $\frac{1}{ip\!\!\!/ + m}$ FFT ($p \leftarrow x$) $ieA\!\!\!/$ FFT ($x \leftarrow p$) ..... $\frac{1}{ip\!\!\!/ + m}$ FFT ($p \leftarrow x$) $ieA\!\!\!/$ FFT ($x \leftarrow p$) $\frac{1}{ip\!\!\!/ + m}$ FFT ($p \leftarrow x$)

diagonal in the momentum space

No integration or linear solver is necessary. **Sequence of multiplying diagonal matrices**.

Very effectively done on computers.

Cost of FFT is also similar to the multiplication of a diagonal matrix, O(N log N).

… Wait a minutes.

$$\frac{\delta S_{\rm f}}{\delta A_\mu(n)} = -\frac{1}{16}\mathrm{Tr}\left(\frac{\delta D}{\delta A_\mu(n)} D^{-1}\right).$$

This is integration.

Do we need to repeat the calculation V=L³T times?

That's practically not possible.

Yes. But, for this, one can use a **stochastic trick**.

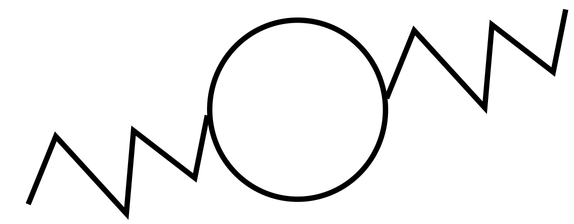
$$\mathrm{Tr}\left(\frac{\delta D}{\delta A_\mu(n)} D^{-1}\right) = \left\langle \zeta^\dagger \frac{\delta D}{\delta A_\mu(n)} D^{-1}\zeta \right\rangle_\zeta ,$$

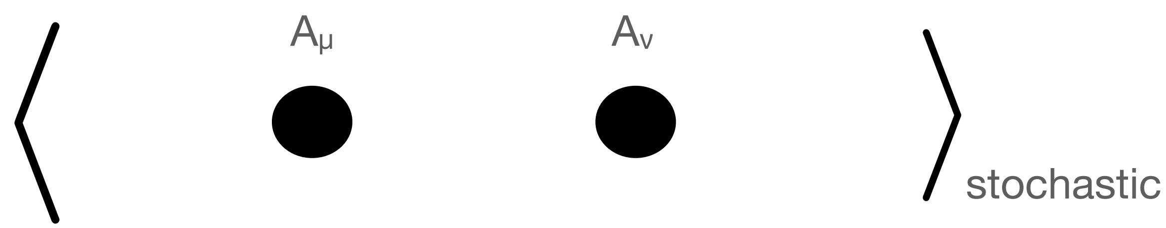
We generate a **random** spinor, ζ(x), with the gaussian weight and take the inner product.

Over many Langevin steps, it averages to the trace.

Now **all fermion loops** are taken care.

Let's calculate photon 2-point functions:





transverse — This part is exact

$$\langle \tilde{A}_\mu(k)\tilde{A}_\nu(k')\rangle = e^{-2\hat{k}^2/\Lambda_{\rm UV}^2}\left[\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)\frac{1}{\hat{k}^2+m_\gamma^2-\Pi(\hat{k}^2)}+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\frac{\xi}{\hat{k}^2+\xi m_\gamma^2}\right] \times (2\pi)^4\delta^4(k+k'),$$

Wave function renormalization:

$$Z_3^{-1}(\hat{k}^2) = e^{2\hat{k}^2/\Lambda_{\rm UV}^2}\left(1-\frac{\Pi(\hat{k}^2)}{\hat{k}^2}\right)$$

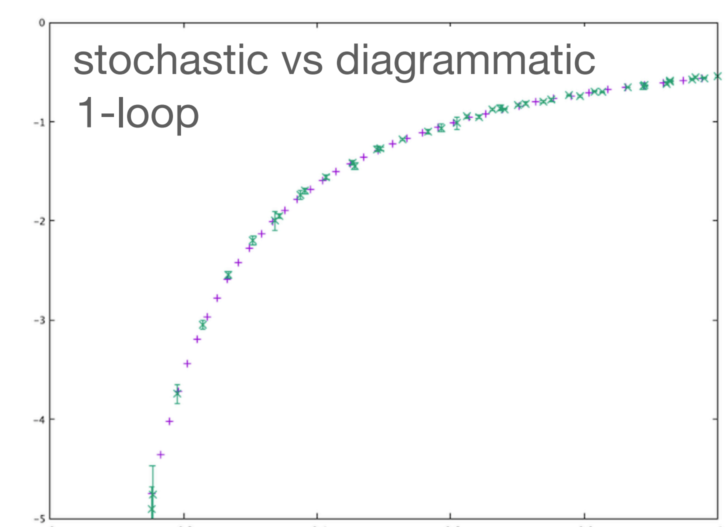
Charge renormalization: $e_{\rm P} = Z_3^{1/2}(0)e,$

Parameter in the Lagrangian

Physical quantity

$$\alpha = \frac{e_{\rm P}^2}{4\pi} \simeq \frac{1}{137}$$

$$e + \frac{Z_3^{(2)}(0)}{2}e^3 + \cdots$$

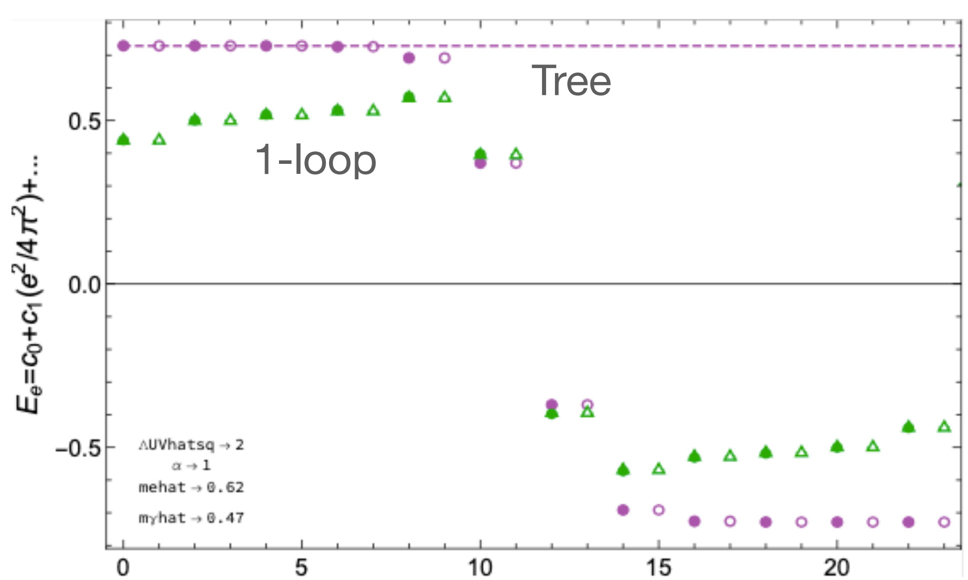


Fermion 2-point functions

$$\left\langle \tilde{D}^{-1}(p,q) \right\rangle = \sum_{n,m} \left\langle (D^{-1})_{nm} \right\rangle e^{-ip\cdot x_n} e^{-iq\cdot x_m}$$

Full propagator: $S(p) \equiv \frac{1}{V} \left\langle \tilde{D}^{-1}(p,-p) \right\rangle$

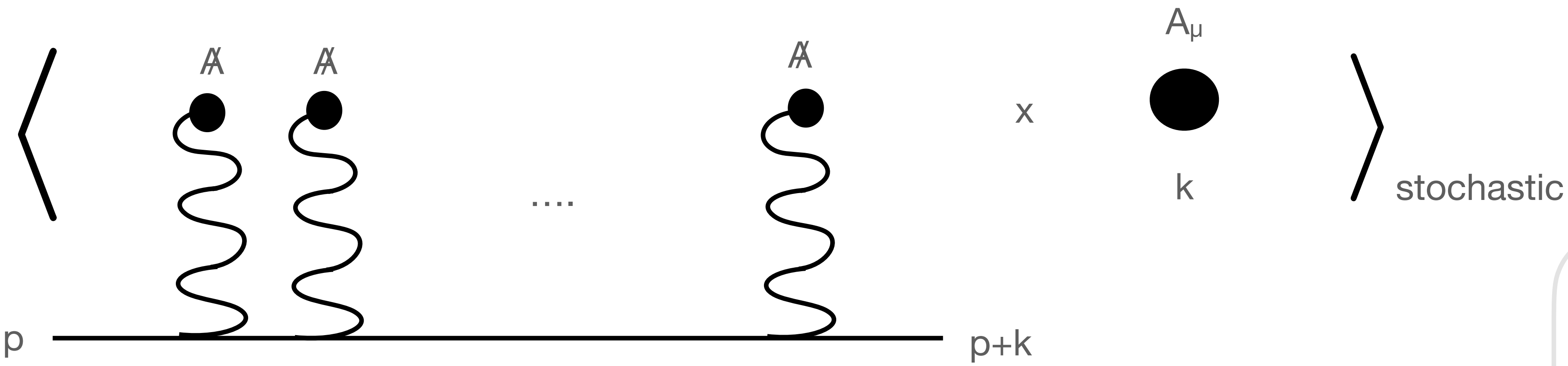
The on-shell fermion energy can be extracted.

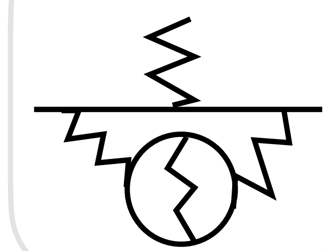


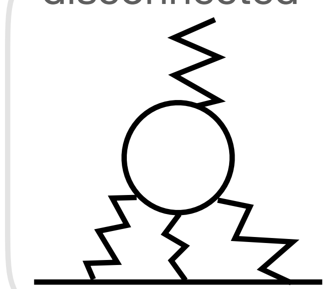
Statistic error not shown but invisible.

Actually, the diagrammatic exact results are overlaid (difference invisible).

3-point functions







$$G_\mu(p,k) = \frac{1}{V}\sum_{n,m,l}\langle\psi(n)\bar{\psi}(m)A_\mu(l)\rangle e^{-ip\cdot x_n}e^{-i(-p-k)\cdot x_m}e^{-ik\cdot(x_l+\mu/2)},$$

$$= \frac{1}{V}\langle\tilde{D}^{-1}(p,-p-k)\tilde{A}_\mu(k)\rangle.$$

3-point function
(All diagrams included.)

Remove external legs. → Vertex function.

full propagators

$$-ie_P\Gamma_\mu(p,k) = \kappa\mathcal{D}^{-1}_{\mu\nu}(k)S(p)^{-1}G_\nu(p,k)S(p+k)^{-1}.$$

Sandwich by wave functions. → Form factors.

$$-ie_P\bar{u}(p)\Gamma_\mu(p,k)u(p+k) = -ie_P\bar{u}(p)\left(F_1(\hat{k}^2)\gamma_\mu - F_2(\hat{k}^2)\frac{\sigma_{\mu\nu}\hat{k}_\nu}{2m_f}\right)u(p+k) + \mathcal{O}(a^2),$$

This part is already done.

g-factor: $$\frac{g}{2} = \frac{F_1(0)+F_2(0)}{F_1(0)}.$$

$(F_1(0) = 1)$

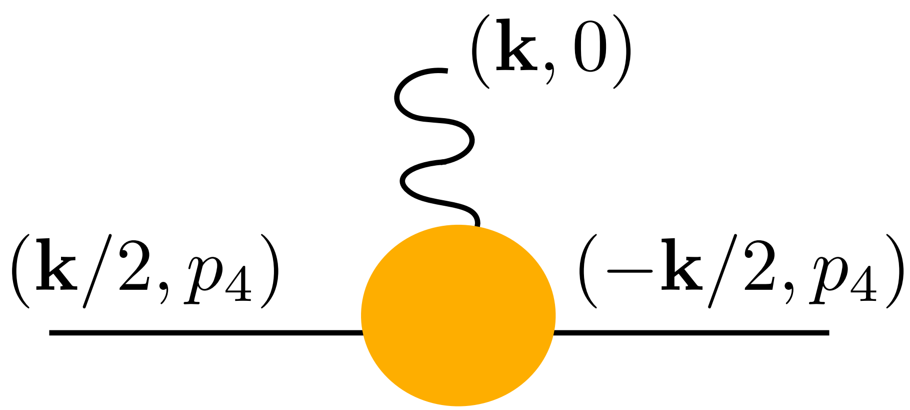
Express this quantity as a **series of** $\frac{\alpha}{\pi} = \frac{e_P^2}{4\pi^2}$. $e_P = Z_3^{1/2}(0)e,$

$$\frac{g}{2} = \frac{g^{(0)}}{2} + \left(\frac{\alpha}{\pi}\right)\frac{g^{(2)}}{2} + \left(\frac{\alpha}{\pi}\right)^2\frac{g^{(4)}}{2} + \cdots$$

Done!

How to get to on-shell?

We are working in the **Euclidean** space. We need **analytic continuation** to the on-shell momentum.

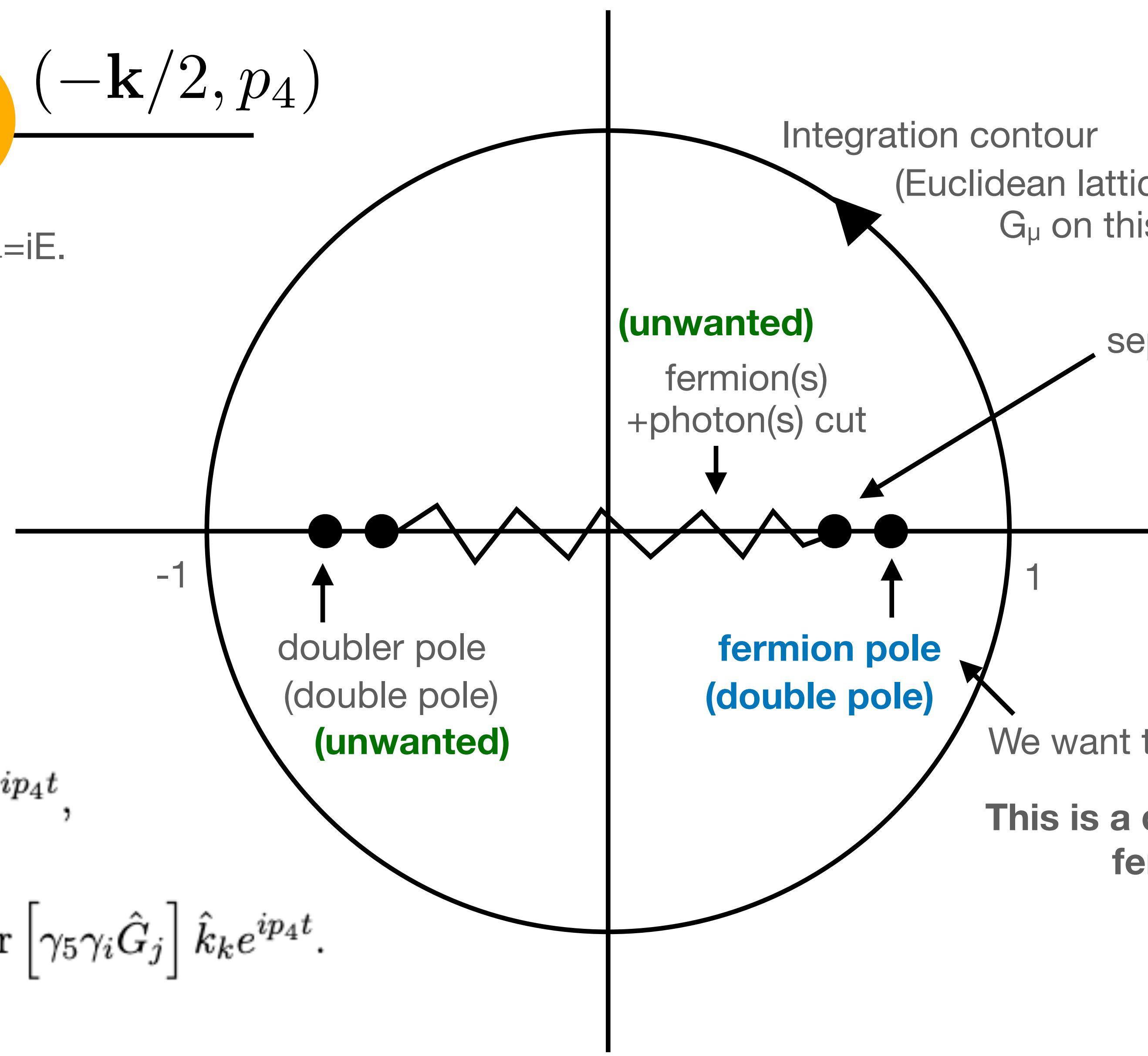


We want 3-point func at p_4=iE.

$$z = e^{ip_4}$$

$$p_4 = 0, \frac{2\pi}{T}, \cdots \frac{2\pi(T-1)}{T}$$

Euclidean energy (periodic b.c.)



Integration contour
(Euclidean lattice calculates
G_μ on this line.)

separated by m_γ

We want the **residue** of this pole.

This is a double pole since both external fermion lines get on-shell.

The continuation can be done by evaluating these integrals:

$$\mathcal{F}_E(t) = \int \frac{dp_4}{2\pi} \mathrm{tr}\left[\gamma_4 \hat{G}_4\right] e^{ip_4 t},$$

$$\mathcal{F}_M(t) = \int \frac{dp_4}{2\pi} \sum_{i,j,k=1}^{3} i\epsilon_{ijk} \mathrm{tr}\left[\gamma_5 \gamma_i \hat{G}_j\right] \hat{k}_k e^{ip_4 t}.$$

The **double pole** gives O(t) terms whose coefficients are proportional to the **on-shell amplitudes**.

$$\mathcal{F}_E(t) = \int \frac{dp_4}{2\pi} \mathrm{tr}\left[\gamma_4 \hat{G}_4\right] e^{ip_4 t},$$

$\propto F_1 + \ldots$

$$\mathcal{F}_M(t) = \int \frac{dp_4}{2\pi} \sum_{i,j,k=1}^{3} i\epsilon_{ijk} \mathrm{tr}\left[\gamma_5 \gamma_i \hat{G}_j\right] \hat{k}_k e^{ip_4 t}.$$

$\propto F_1 + F_2 + \ldots$

Take the double ratio:

$$\frac{g(t)}{2} = \frac{\mathcal{F}_M(t)/\mathcal{F}_E(t)}{\mathcal{F}_M^{(\mathrm{norm})}(t)/\mathcal{F}_E^{(\mathrm{norm})}(t)},$$

$t \to \infty$ of this quantity is the g-factor!

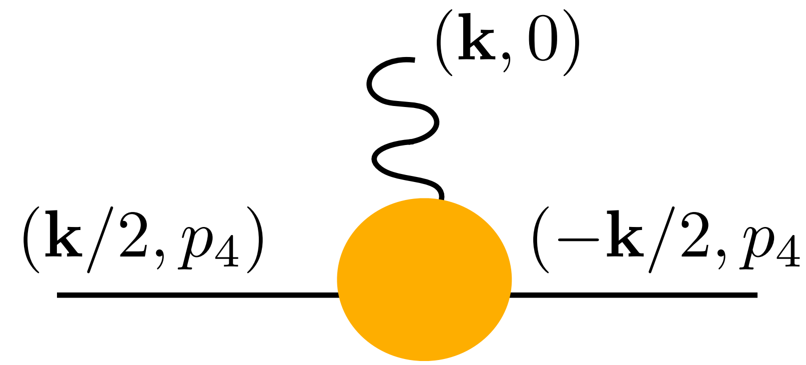
(Here, (norm) denotes the tree level form factors.)

Unwanted terms are suppressed by 1/t and/or exp(-m_γt).

Doubler contributions **cancel** in the ratio.

(At even t, F_E and F_M vanish due to doublers, and doubled at odd t.)

Parameter choices



We have many parameters, and we need to take various limits.

mass² scale (unit 1/a)

Our choice:

	Our choice		
Should be large, but shouldn't be too large to suppress UV div. →	1	$\Lambda_{\rm UV}^2/2$	
		$\gg$	**Large UV cut-off**
↑ Just took the value in between ↓ →	$\sin\frac{\pi}{L}$	m^2	
		$\gg$	**Non-relativistic** fermion momentum
Smallest lattice momentum² →	$\sin^2\frac{\pi}{L}$	$\lvert\hat{\mathbf{k}}\rvert^2/4$	
		$\gg$	**Small photon mass**
Shouldn't be too small to suppress IR div. →	$\sin^3\frac{\pi}{L}$	$m_\gamma^2/4$	

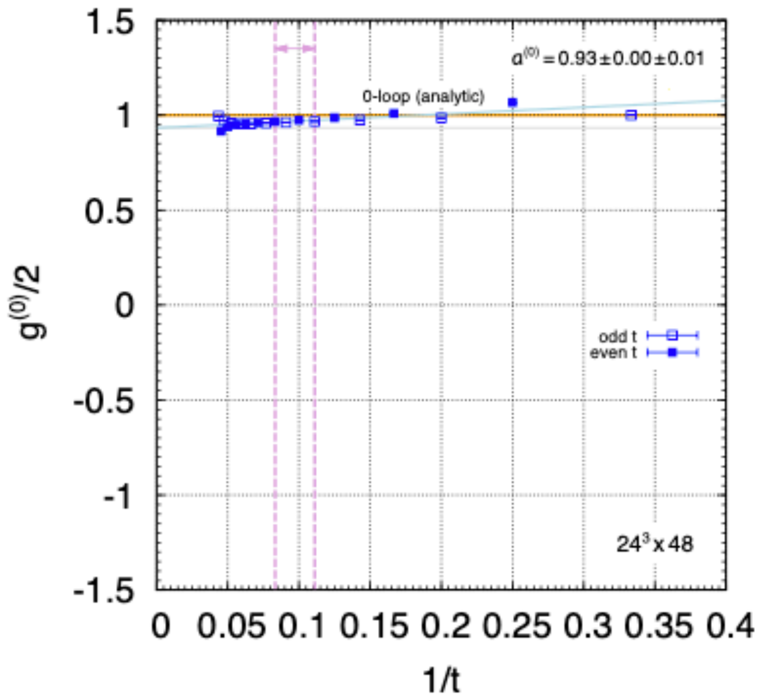
By this choice, **L→∞ limit is the continuum theory** with the massless photon. Errors are of $O\left(\frac{\pi}{L}\right)$

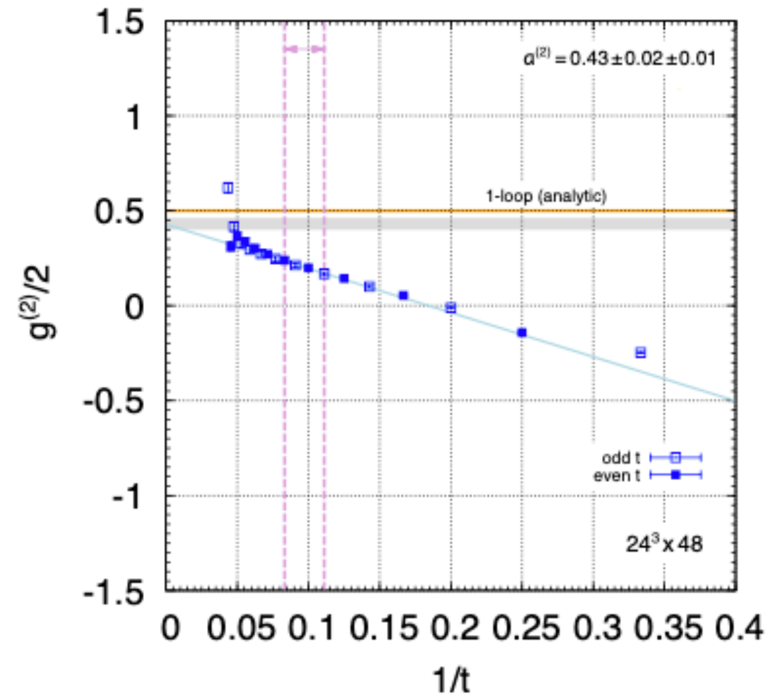
O(a few - 10%) in the realistic simulations.

24^3x48 lattice results:

$$\frac{g(t)}{2} = \frac{g^{(0)}(t)}{2} + \frac{g^{(2)}(t)}{2}\left(\frac{\alpha}{\pi}\right) + \frac{g^{(4)}(t)}{2}\left(\frac{\alpha}{\pi}\right)^2 + \frac{g^{(6)}(t)}{2}\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

$L^3 \times T$	ma	$(\Lambda_{\rm UV}a)^2$	ξ	$m_\gamma a$	ϵ	$N_{\rm conf}$
$12^3 \times 24$	0.51	2.0	1.0	0.26	0.02	4800
$16^3 \times 32$	0.44	2.0	1.0	0.17	0.02	6400
$20^3 \times 40$	0.39	2.0	1.0	0.12	0.02	7040
$24^3 \times 48$	0.36	2.0	1.0	0.094	0.02	9600

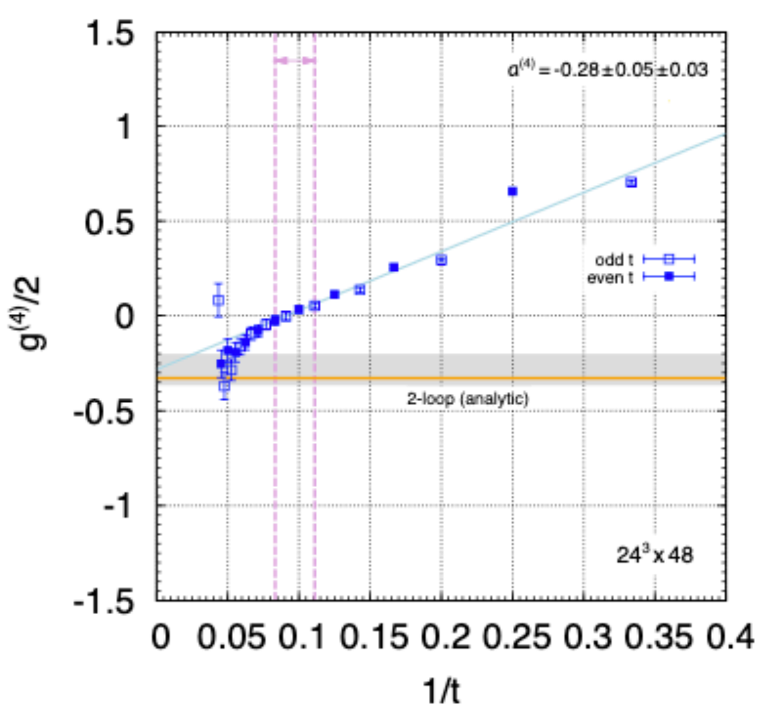


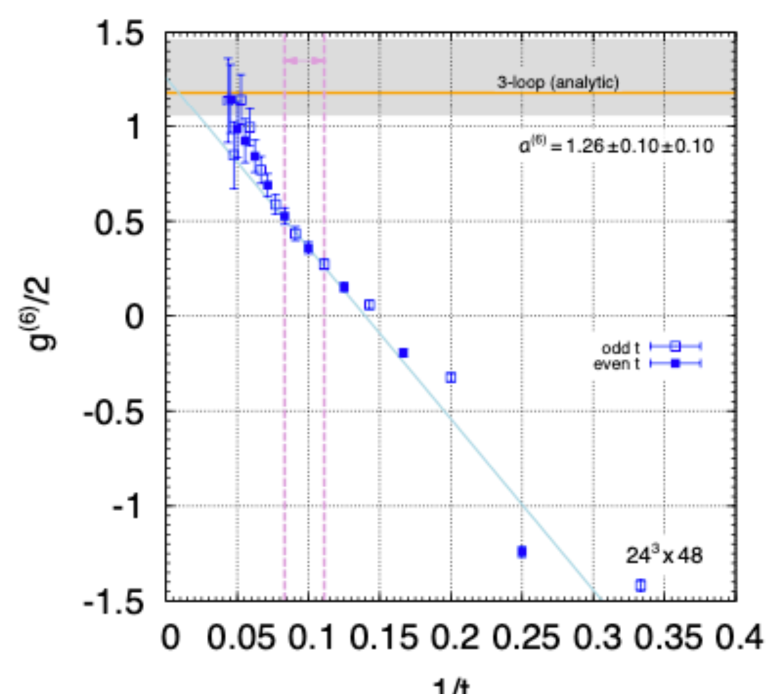


$$\frac{g(t)}{2} = \frac{\mathcal{F}_M(t)/\mathcal{F}_E(t)}{\mathcal{F}_M^{(\rm norm)}(t)/\mathcal{F}_E^{(\rm norm)}(t)},$$

$$\simeq a^{(2n)} + b^{(2n)}/t,$$

We see the "a + b/t" behavior.



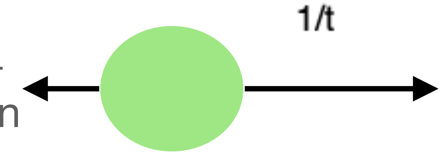


Extrapolating

$$t \in [L/2-3, L/2]$$

we get gray shaded values at each order.

Finite volume + IR contamination



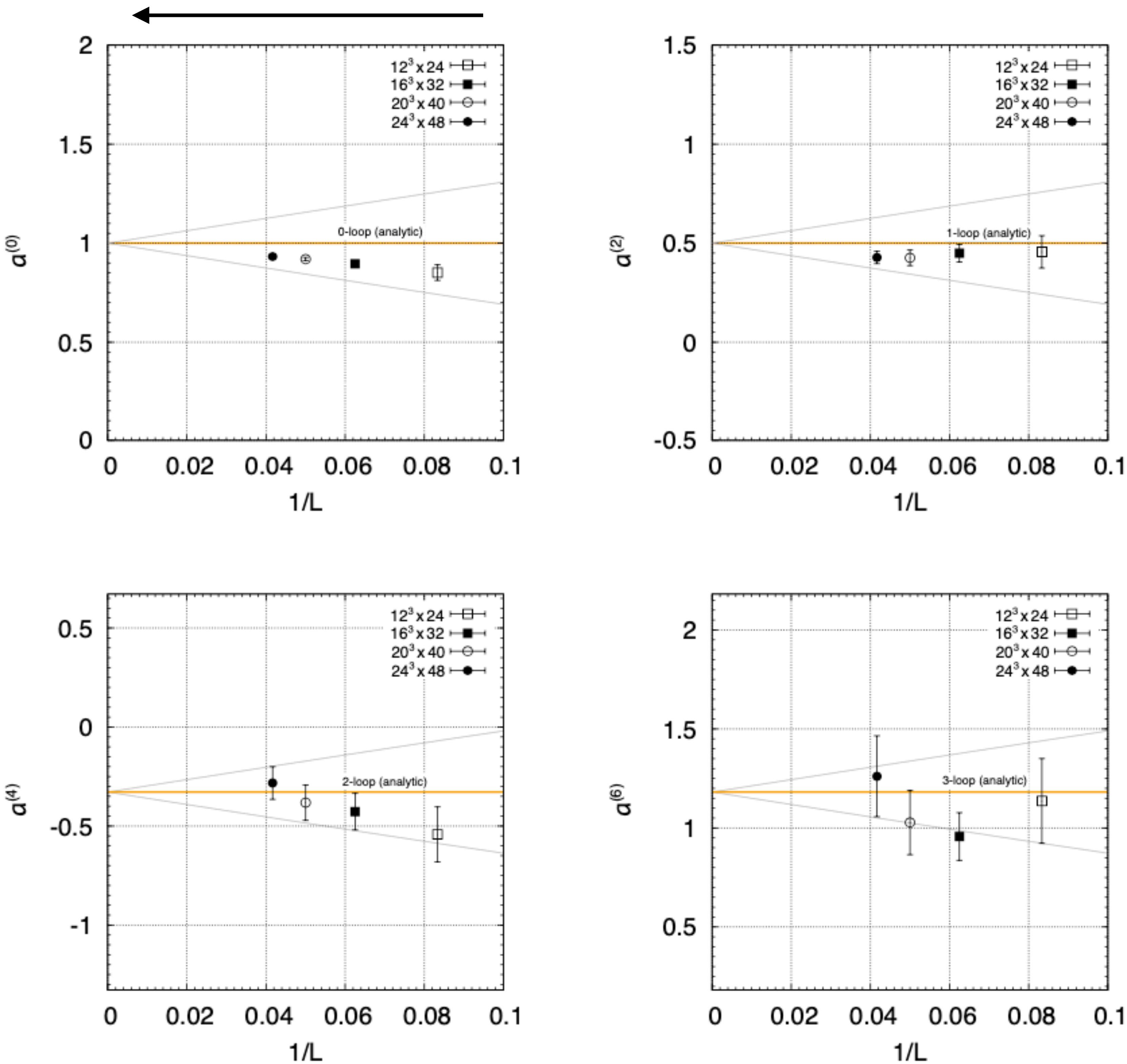
cut contribution

We should use this region for extrapolation to large t.

L dependence

Remember that L→∞ limit is the continuum theory.

Continuum, infinite volume, infinite UV cut-off, zero photon mass



Well, may be good enough for the first try.

$L^3 \times T$	ma	$(\Lambda_{\mathrm{UV}}a)^2$	ξ	$m_\gamma a$	ϵ	N_{conf}
$12^3 \times 24$	0.51	2.0	1.0	0.26	0.02	4800
$16^3 \times 32$	0.44	2.0	1.0	0.17	0.02	6400
$20^3 \times 40$	0.39	2.0	1.0	0.12	0.02	7040
$24^3 \times 48$	0.36	2.0	1.0	0.094	0.02	9600

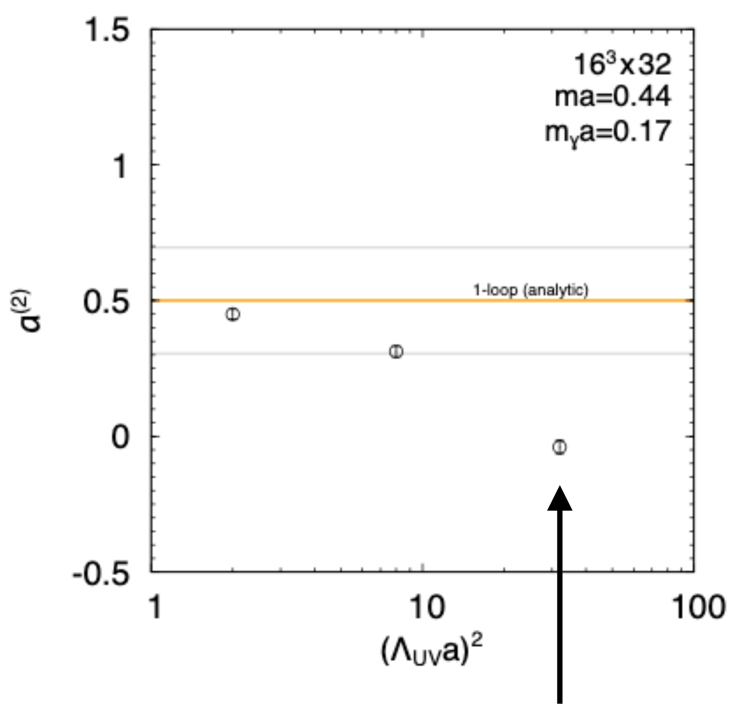
$L^3 \times T$	$a^{(0)}$	$a^{(2)}$	$a^{(4)}$	$a^{(6)}$
$12^3 \times 24$	0.85(0)(4)	0.46(2)(6)	−0.54(2)(11)	1.14(4)(17)
$16^3 \times 32$	0.90(0)(2)	0.45(2)(2)	−0.43(3)(6)	0.96(6)(6)
$20^3 \times 40$	0.92(0)(1)	0.43(2)(2)	−0.38(4)(5)	1.03(8)(8)
$24^3 \times 48$	0.93(0)(1)	0.43(2)(1)	−0.28(5)(3)	1.26(10)(10)
(analytic)	1	0.5	−0.328 . . .	1.18 . . .

Tone down a little bit….

There are a few problems which we need to take care in future works.

1. Log divergences

The result actually has a large Λ_{UV} dependence.



This is mainly due to **log divergence** in the mass renormalization.

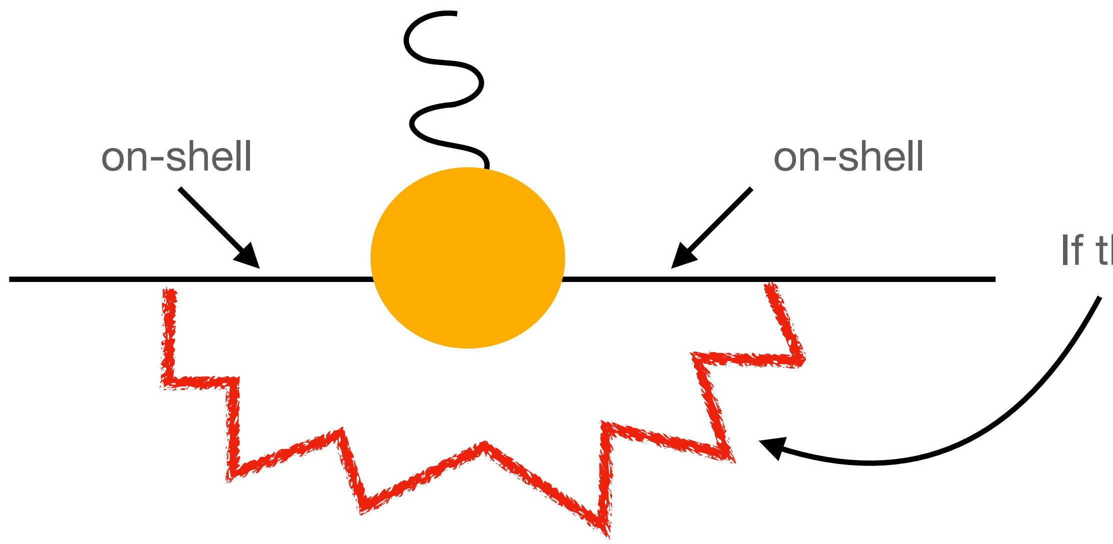
If we didn't introduce Λ_{UV}, the continuum limit gets **far away**.

One should carefully choose UV cut-off. Or, maybe we need some improved methods.

2. IR divergences

Finite m_γ regulates the **IR divergence**, but we need to make sure there is no $1/m_\gamma$ effects in the final formula.

There is a little bit of complication in a finite volume.



If this photon momentum is zero, we have

$$\frac{1}{V}\frac{1}{m_\gamma^2}$$ Photon propagator (1/V suppressed by the measure.)

$$\left(\frac{1}{s_4(p)^2+E^2}\right)^2$$ Two extra on-shell fermion propagators

This gives $O\left(\frac{t^3}{m_\gamma^2 V}\right)$ contributions to F_E and F_M. This is much **larger** than leading O(t) effects for a large t.

(soft theorem)

In fact, those are **largely cancelled in the ratio F_M/F_E** just as in the continuum theory, but not exactly in a **finite volume**.

The effects are small enough in the current O(10%) measurements, but will be important when we need more accuracy.

(Not so serious, though. We can fit and remove the IR contributions.)

3. Finite volume effects

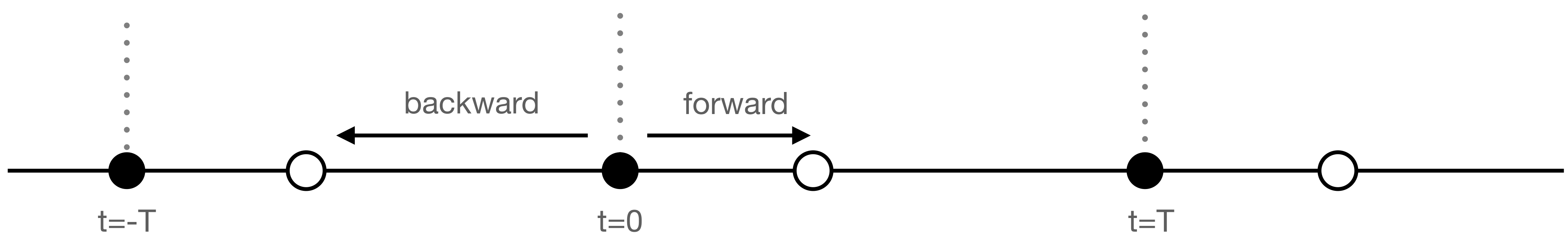
The accuracy is already 1/L. Yes, finite volume effect is large.

There are actually **annoying** effects in the **perturbation theory in a finite volume**.

We take the **periodic** boundary conditions for the fermion.

$$\frac{g(t)}{2} = \frac{\mathcal{F}_M(t)/\mathcal{F}_E(t)}{\mathcal{F}_M^{(\mathrm{norm})}(t)/\mathcal{F}_E^{(\mathrm{norm})}(t)},$$

We want to see the large t behavior, but this function is **periodic**.



Correlation functions have a contribution from the **backward propagation**.

For example,

leading IR divergence

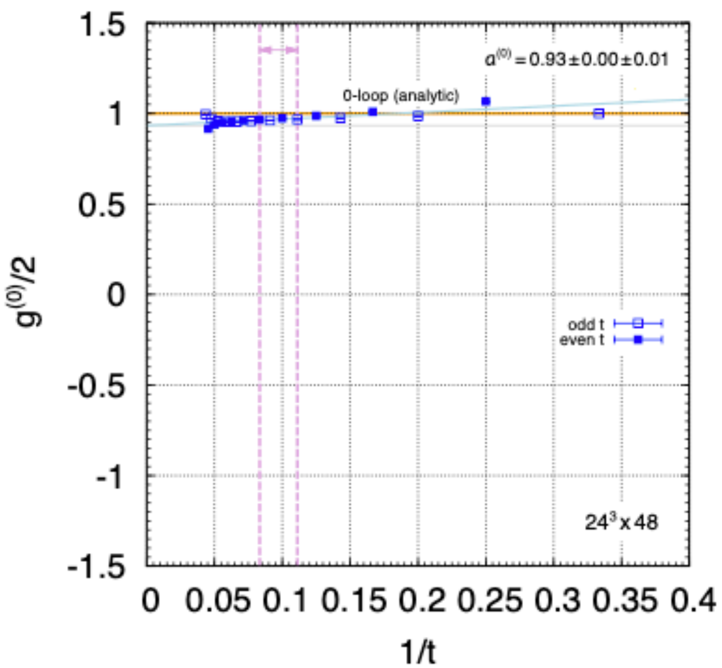
backward propagation

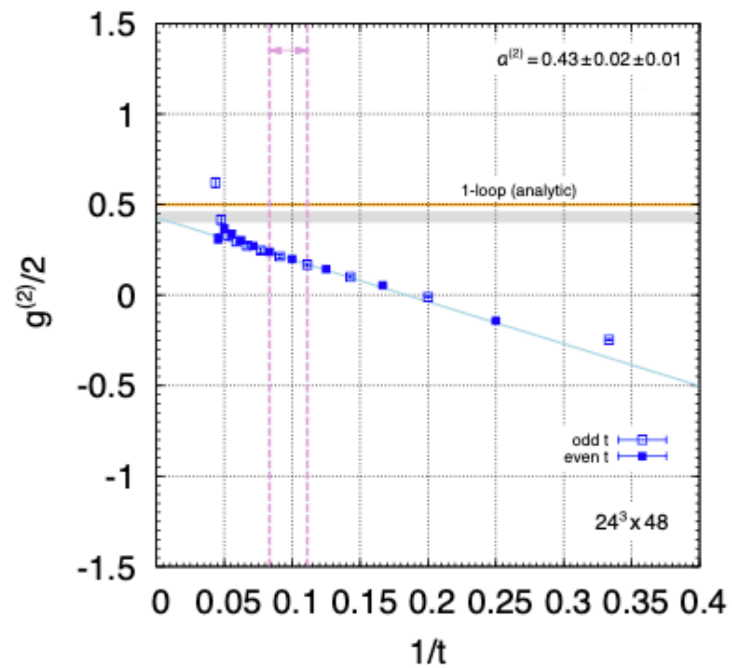
$$\begin{aligned}\mathcal{F}_E(t) &\sim t z_*^t e^{-xt^2} + (T-t) z_*^{(T-t)} e^{-x(T-t)^2} \\ &\sim t e^{-Et} e^{-xt^2}\left[1 + \left(\frac{T}{t}-1\right) e^{-E(T-2t)} e^{-xT(T-2t)}\right] \\ &= t e^{-Et} e^{-xt^2}\left[1 + \left(\frac{T}{t}-1\right) e^{-E_0(T-2t)}\left(1 - e^2\left(E_1 + \frac{1}{2L^3 m_\gamma^2}\right)(T-2t) + \cdots\right)\right].\end{aligned}$$

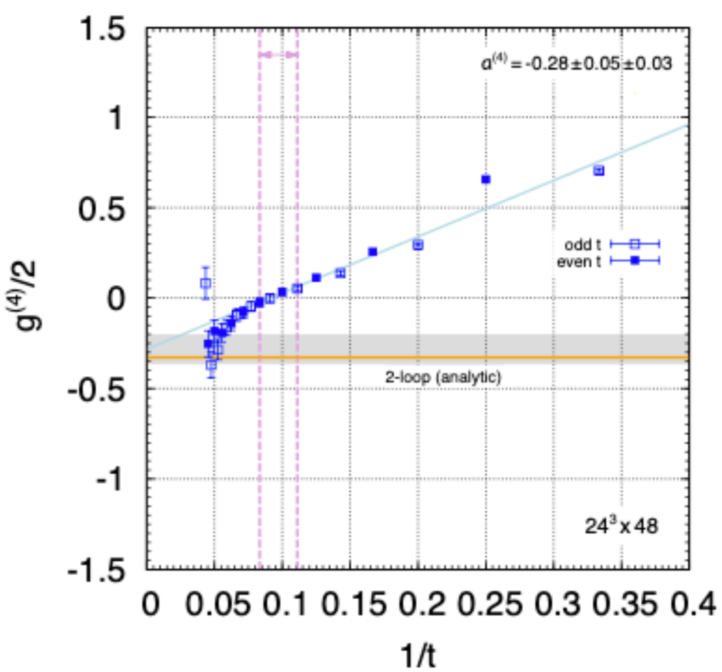
This part cancels in the ratio.

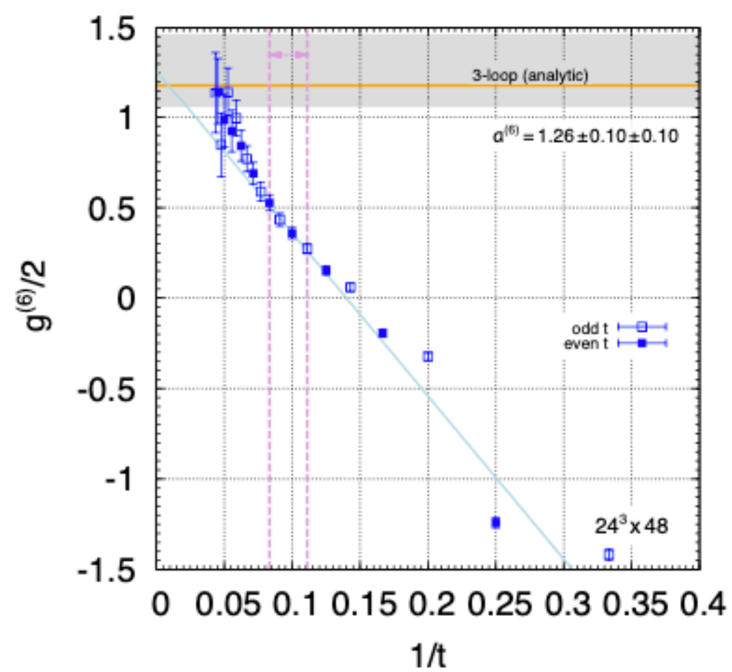
The contribution gives **higher power of t** in the higher order in the perturbations. It gets important at large t.

We already see such a tendency: Deviation from the straight line gets larger for higher orders.





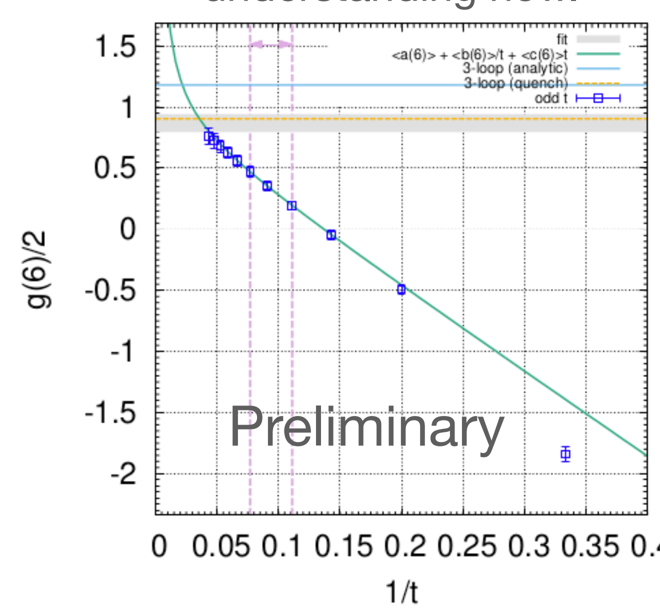




This is going to be an obstacle for higher order computations.

It turns out, manipulating the fermion boundary condition can avoid this problem.

We actually have better understanding now.



5-loop no lepton loop (preliminary)

[Volkov '19]

both Monte Carlo integration and analytical calculations. For example, the uncertainty in (2) is entirely determined by that contribution. Also, it is the contribution that suffered the most from found mistakes and corrections; see Ref. [34]. The value

$$A_1^{(10)}[\text{no lepton loops, AKN}] = 7.668(159). \quad (3)$$

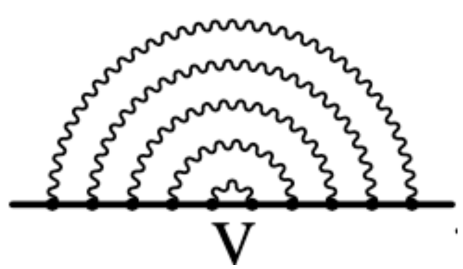
can be obtained by using (2) and the value of the remaining part that can be extracted from Ref. [34]. By 2019, there was no independent calculations of $A_1^{(10)}$[no lepton loops].

We recalculated this contribution with the help of the supercomputer "Govorun" (JINR, Dubna, Russia). 40000 GPU-hours of Monte Carlo integration on NVidia Tesla V100 that were spread over several months have led to the result

$$A_1^{(10)}[\text{no lepton loops, Volkov}] = 6.793(90), \quad (4)$$

where the uncertainty corresponds to 1σ limits. It is in good agreement with the preliminary value 6.782(113) published in Ref. [35]. The descrepancy between this result and (3) is approximately 4.8σ. This means that the values are probably different. The reason of this difference is unknown. Sec.

[Aoki, Kinoshita, Nio '19]



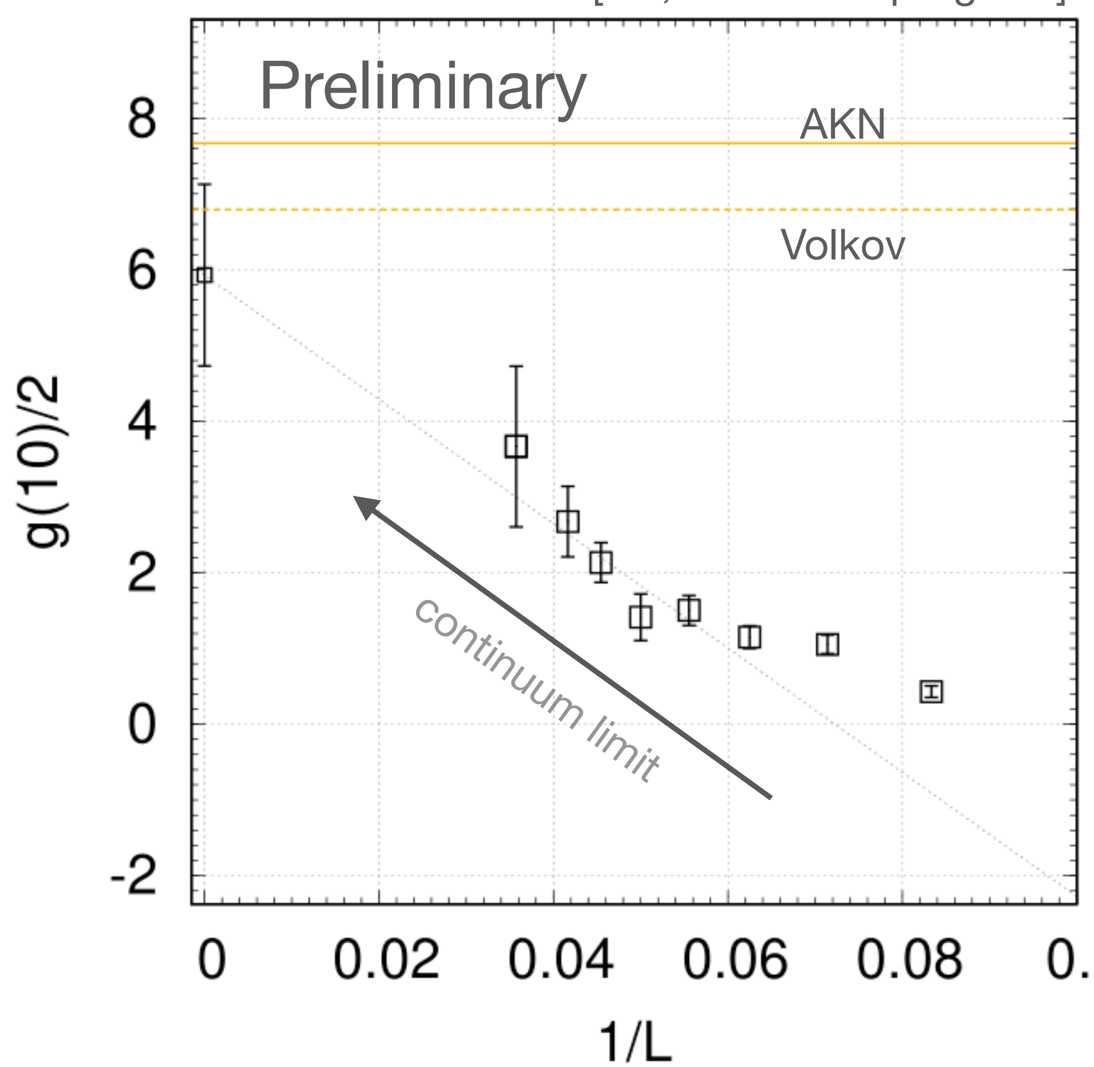
6354 diagrams

The discrepancy is in the subset of diagrams with **no lepton loops** → quenched QED.

Photon is a free field → Langevin evolution not necessary.

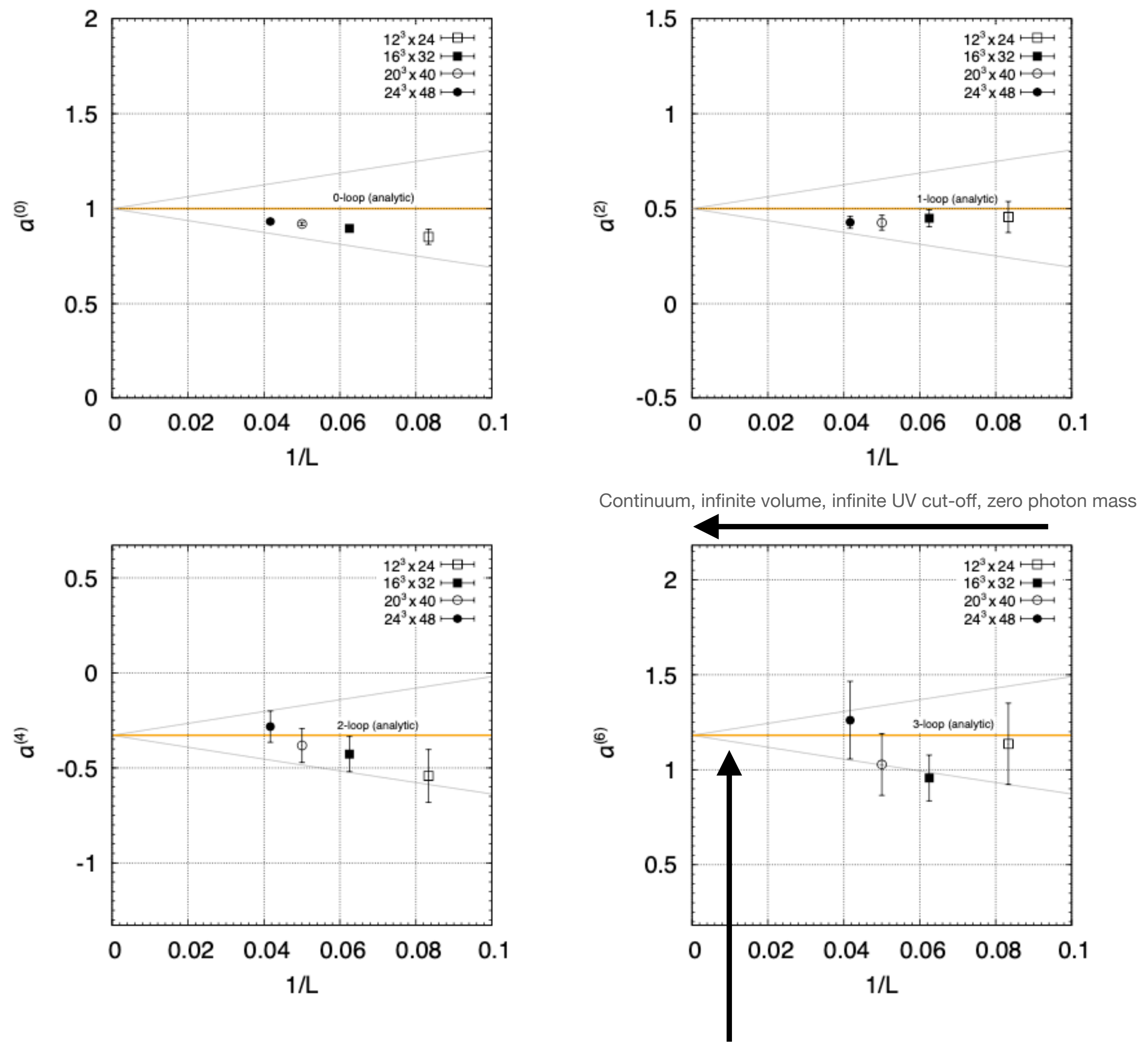
Very easy!

[RK, Takaura in progress]



We need larger volumes for more precise measurements, but looks doable.

Summary



I tried. It worked up to 3-loops with O(10%) accuracy.

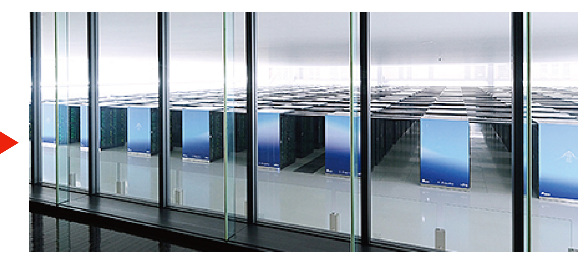
Maybe useful.

I don't find a very serious problem to go to higher loops.

(Computational cost scales as Volume*(loop)^2)

We used NEC SX-Aurora TSUBASA A500 at KEK.

?



Fugaku (wikipedia)

We expect a currently available larger scale computer with a more optimized code will make larger lattice volumes such as **L=128** possible.

L~100 ← A few percent level measurements will be possible(?)

Anyway, the stochastic method works for **physical quantities** like g-2.

If people want 6-loops, this may be a good method for the first estimation.

Maybe useful for a wider class of physical quantities in general theories.