

Boundary Scattering and Non-Invertible Symmetries in 1+1 Dimensions

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Based on S. Shimamori, SY, arXiv:2504.08375

Introduction

Non-invertible symmetry

A class of generalized symmetry “Does not have inverse”

A nice review [Shao, 2308.00747]

Found in many QFTs:

2 dim: [Verlinde], ..., [Frohlich, Fuchs, Runkel, Schweigert],... [Bhardwaj, Tachikawa],
[Chang, Lin, Shao, Wang, Yin],...

Higher dim:

[Koide, Nagoya, SY], [Kaidi, Ohmori, Zheng], [Choi, Cordova, Hsin, Lam, Shao],...

Any application ?

[Copetti, Cordova, Komatsu 24]

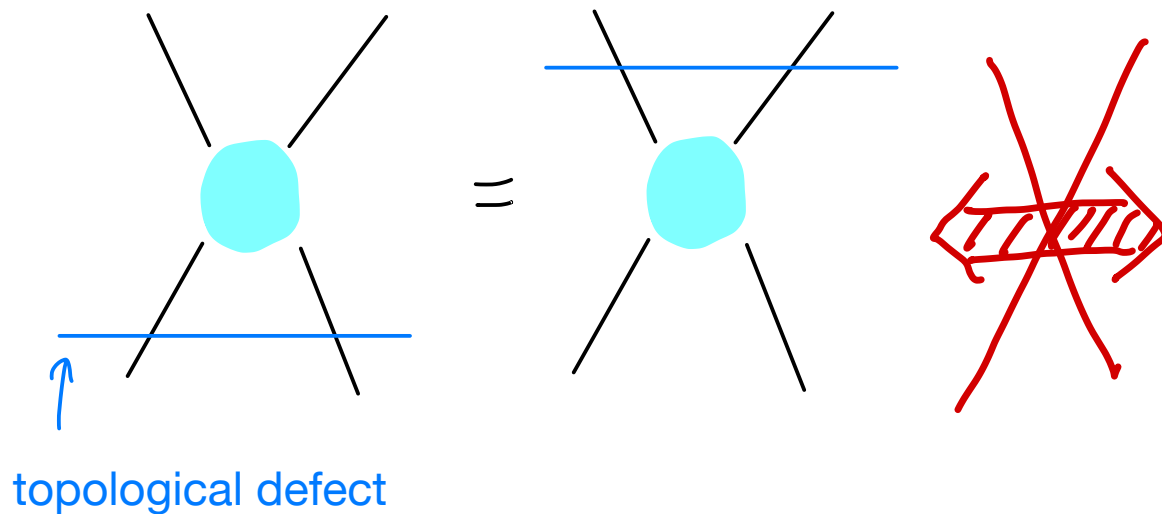
Application for

Scattering Amplitude

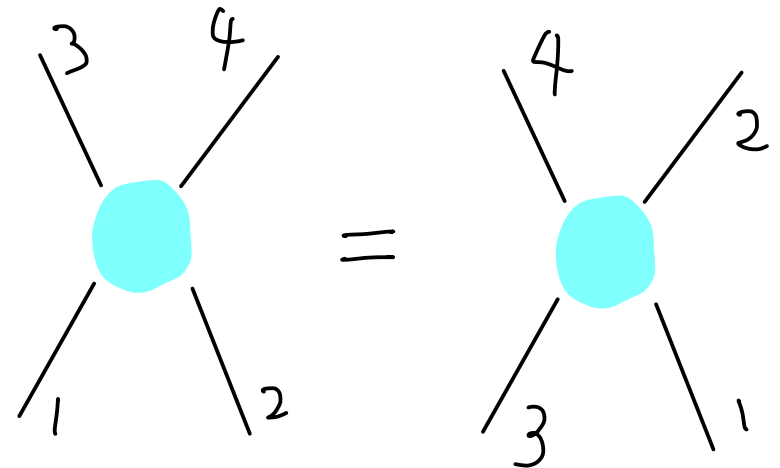
2D integrable system with non-invertible symmetry

Assume unitarity and Yang-Baxter equation

Ward-Takahashi identity for non-invertible symmetry



Crossing symmetry



Incompatible!

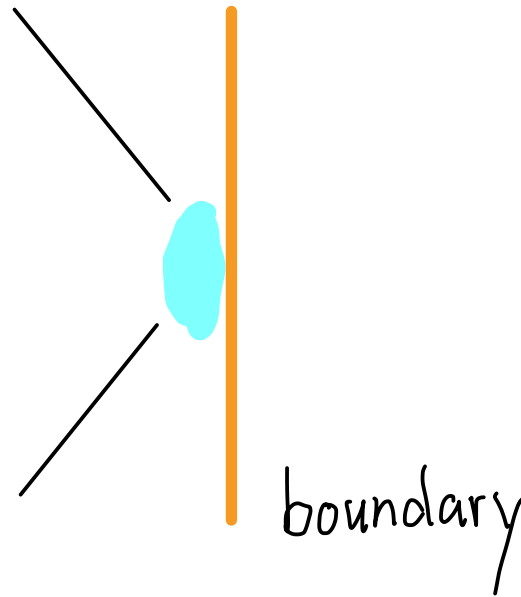
Crossing symmetry must be modified !

Old: $S_{dc}^{ab}(\theta) = S_{ad}^{bc}(i\pi - \theta)$

New: $S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$

Detail will be explained later.

[Shimamori, SY] considered boundary scattering with non-invertible symmetry.

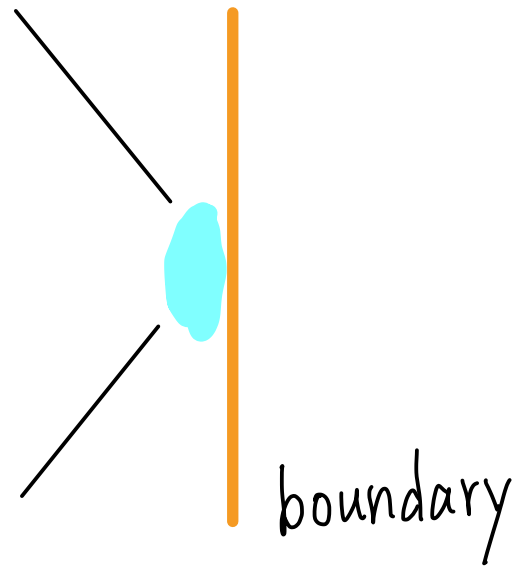


Why boundary?

My long term interest:

- D-branes in string theory (boundary of worldsheet).
- Understanding anomalies.
- ...

Result [Shimamori, SY]



Boundary S-matrix

$$\Rightarrow R_{ca}^b(\theta)$$

Old:

$$R_{ca}^b\left(\frac{i\pi}{2} - \theta\right) = \sum_d S_{cd}^{ba}(2\theta) R_{ca}^d\left(\frac{i\pi}{2} + \theta\right)$$

New:

$$R_{ca}^b\left(\frac{i\pi}{2} - \theta\right) = \sum_d \sqrt{\frac{d_d}{d_b}} S_{cd}^{ba}(2\theta) R_{ca}^d\left(\frac{i\pi}{2} + \theta\right)$$

Why crossing symmetry is modified?

S-matrix

Reconsider !

LSZ formula

Crossing symmetry

LSZ formula fails to be true in some cases.

Plan

- LSZ formula
- LSZ formula fails?
- SSB of non-invertible symmetry and modified LSZ formula
- Boundary scattering
- Summary and discussion

LSZ formula

Outline

Particle

elementary, composite, soliton,...

State (One particle state)

$|k, i\rangle$
↑ momentum ↑ other label



S-matrix

Fields

elementary, composite, may not be written as a functional of elementary fields,...

Operator

$\phi(x)$



Correlation function



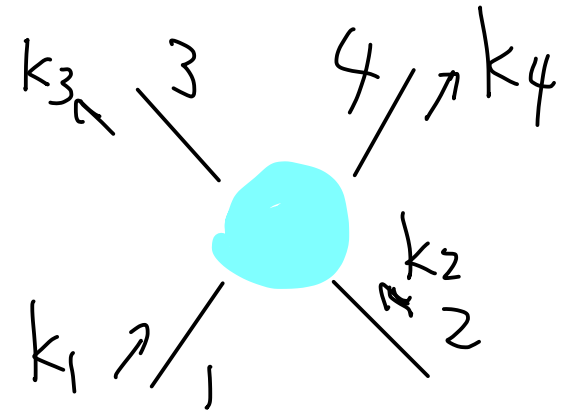
LSZ formula
(Non-perturbative !)

Setup

d-dim, scalar particles, 1, 2, 3, 4

elementary, composite, soliton, ... m_1, m_2, m_3, m_4

scattering $12 \rightarrow 34$



S-matrix element

$$S_{12 \rightarrow 34}(s, t)$$

$$s := -(k_1 + k_2)^2$$
$$t := -(k_1 - k_3)^2$$

Assumption

$$\exists \phi_i(x) \\ i = 1, 2, 3, 4$$

Local operators

elementary, composite, may not be written as a functional of elementary fields,...

s.t.

$$\langle k_i, i | \phi_i(x) | 0 \rangle \neq 0$$

(This condition is satisfied unless you have any reason for vanishing.)

LSZ formula. (See eg. [Peskin, Schroeder Ch7])

$$\phi_i(x) \xrightarrow{\text{Fourier}} \tilde{\phi}_i(k)$$

2pt function

$$\langle \tilde{\phi}_i(k) \tilde{\phi}_i(k') \rangle = \Delta_i(k^2) \frac{1}{i} (2\pi)^d \delta^d(k+k')$$

$$\Delta_i(k^2) \underset{\text{pole}}{\sim} \frac{R_i}{k^2 + m_i^2} \quad (R_i > 0)$$

$$S_{12 \rightarrow 34}(s, t) = \lim_{\substack{\text{on-shell} \\ k_i^2 \rightarrow -m_i^2}} \frac{i\sqrt{R_1}}{\underbrace{\Delta_1(k_1^2)}} \frac{i\sqrt{R_2}}{\underbrace{\Delta_2(k_2^2)}} \frac{i\sqrt{R_3}}{\underbrace{\Delta_3(k_3^2)}} \frac{i\sqrt{R_4}}{\underbrace{\Delta_4(k_4^2)}_{\text{"amputate"}}}$$

$$\times \langle \tilde{\phi}_1(-k_1) \tilde{\phi}_2(-k_2) \tilde{\phi}_3(k_3) \tilde{\phi}_4(k_4) \rangle$$

Crossing symmetry

$$S_{12 \rightarrow 34}(s, t) = \lim_{\substack{\text{on-shell} \\ k_i^2 \rightarrow -m_i^2}} \frac{i\sqrt{R_1}}{\Delta_1(k_1^2)} \frac{i\sqrt{R_2}}{\Delta_2(k_2^2)} \frac{i\sqrt{R_3}}{\Delta_3(k_3^2)} \frac{i\sqrt{R_4}}{\Delta_4(k_4^2)} \langle \tilde{\phi}_1(-k_1) \tilde{\phi}_2(-k_2) \tilde{\phi}_3(k_3) \tilde{\phi}_4(k_4) \rangle$$

Invariant under exchange:

$$2 \leftrightarrow 3, \quad k_2 \leftrightarrow -k_3 \quad (\Rightarrow \quad s \leftrightarrow t)$$

$\Downarrow$

Crossing symmetry

$$S_{12 \rightarrow 34}(s, t) = S_{13 \rightarrow 24}(t, s)$$

(Analytically continued)

LSZ formula fails?

Assumption: $\exists \phi_i(x)$ s.t. $\langle k, i | \phi_i(x) | 0 \rangle \neq 0$

$\Updownarrow$ Not satisfied

$$\forall \phi_i(x) : \text{local operator}, \quad \langle k, i | \phi_i(x) | 0 \rangle = 0$$

Does this happen ?

Yes !

Example

1+1 dim, finite global symmetry is spontaneously broken,

$\Rightarrow$ **Kink solitons** satisfy

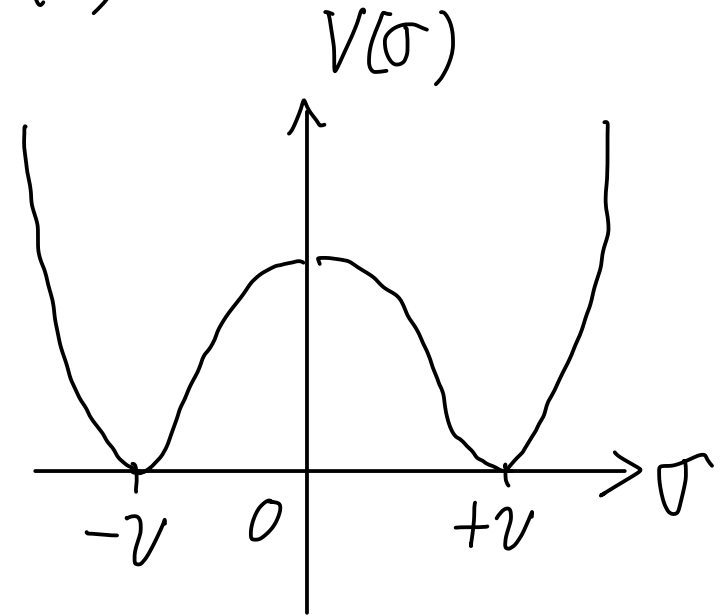
$$\forall \phi_i(x) : \text{local operator} , \quad \langle k, i | \phi_i(x) | 0 \rangle = 0$$

Example

(1+1) dim real scalar

$\sigma(x)$

$$S = \int d^2x \left[-\frac{1}{2} (\partial\sigma)^2 - V(\sigma) \right]$$



$\mathbb{Z}_2$ Symmetry η

$$\hat{\eta} \hat{\sigma}(x) \hat{\eta}^{-1} = -\hat{\sigma}(x)$$

spontaneously broken $\Rightarrow$

Two vacua $|+v\rangle, |-v\rangle$

$$\hat{\eta} |-v\rangle = |+v\rangle$$

$$\hat{\eta} |+v\rangle = |-v\rangle$$

time
 $\uparrow$

$\hat{\eta}$

$+v$

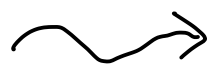
$-v$

Symmetry $\Rightarrow$ Topological defect

$$\hat{H} \hat{\eta} = \hat{\eta} \hat{H}$$

$$\eta \text{ —————}$$

$$\begin{array}{c} \text{time} \\ \uparrow \\ = \eta \text{ —————} \end{array}$$



invariant under continuous deformation

$$\eta \text{ —————} = \text{~~~~~} \eta$$

An explanation:

In curved space-time, no canonical time slice, but η is conserved.

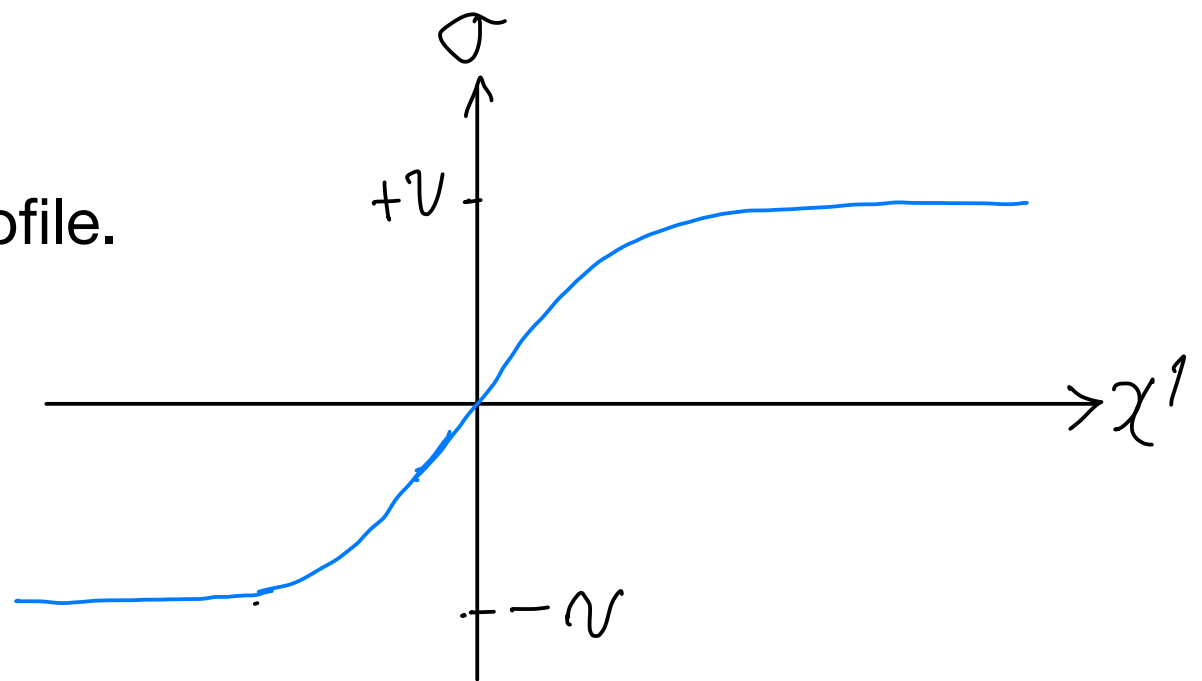
$\Rightarrow$ Invariant in any time slice.

Kink soliton

Static, finite energy $\sigma(x)$ profile.

$$x^1 \rightarrow -\infty, \quad \sigma(x) \rightarrow -V$$

$$x^1 \rightarrow +\infty, \quad \sigma(x) \rightarrow +V$$



Smallest energy state = stable

Kink (soliton)

- a discrete symmetry is spontaneously broken, kink solitons can exist.
- In 1+1 dim, a kink soliton is a particle.

A kink soliton does not satisfy an assumption of LSZ formula, i.e.

$$|0\rangle := |-v\rangle$$

$\forall \phi(x)$ local operator

$$\langle \text{kink} | \phi(x) | 0 \rangle = 0$$

⊙ $\phi(x)$ cannot change $|0\rangle = |-v\rangle$ in $x^1 \rightarrow \infty$

$\langle \text{kink} |$

$-v \quad \not\equiv \quad +v$

↑ overlap vanish

$|0\rangle$

$\times$
 $\phi(x)$

$-v$

$-v$

$\longrightarrow x^1$

Any alternative?

“Non-genuine local operator”

$\phi(x)$

x

γ ← topological defect
of the symmetry

Non-local, but non-local part is topological.

$-\nu$ $+\nu$
 $\phi(x)$
 x

← Includes kink soliton

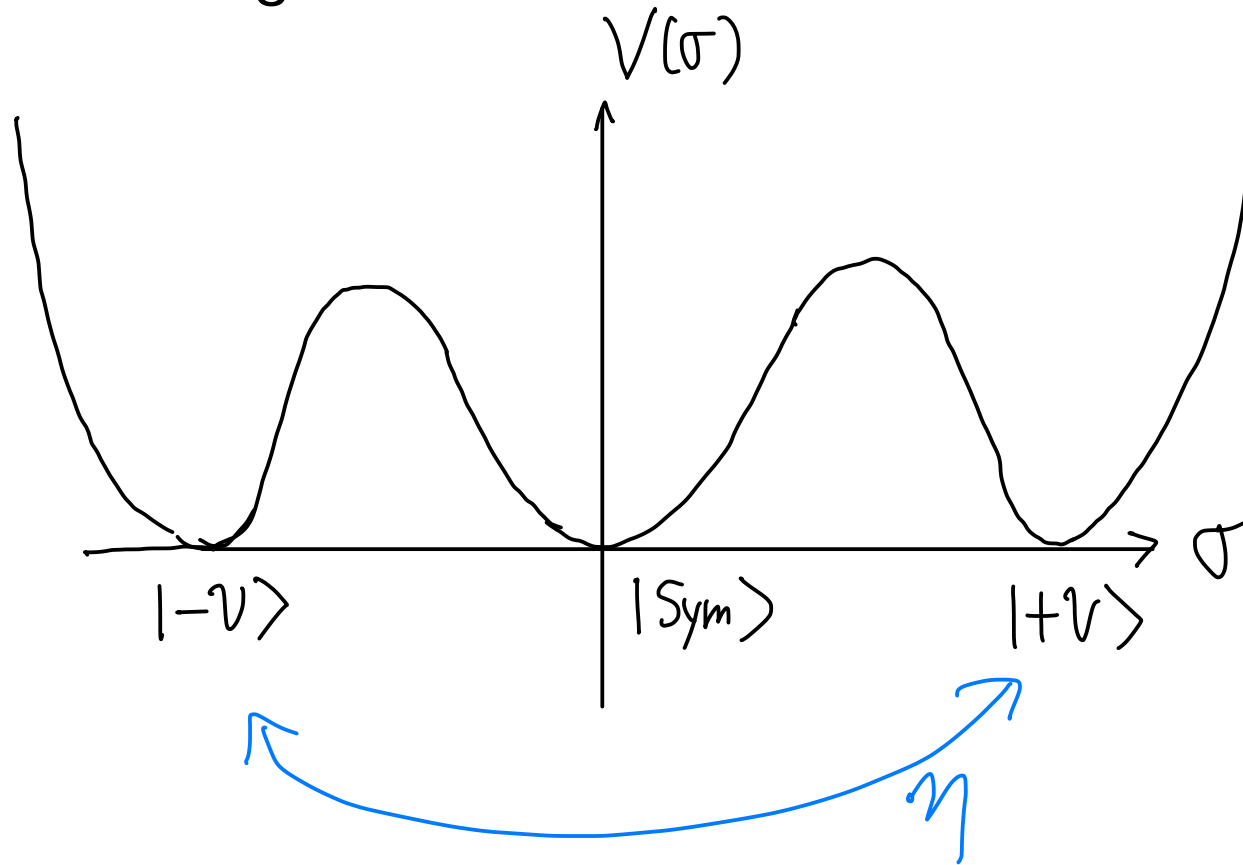
$|0\rangle$ $-\nu$ $-\nu$

$\Rightarrow$ reconsider LSZ formula

SSB of non-invertible symmetry
and modified LSZ formula

Example: "tricritical Ising CFT, gapped deformation by $\phi_{1,3}$

Skematic figure:



3 degenerate
vacua

How about $|Sym\rangle$?

It cannot explained by a conventional symmetry.

This degeneracy can be explained “non-invertible symmetry.”

Operators $1, \hat{\eta}, \hat{\mathcal{N}}$

“Ising fusion algebra”

$$\hat{\eta}^2 = 1$$

$$\hat{\mathcal{N}} \hat{\eta} = \hat{\eta} \hat{\mathcal{N}} = \hat{\mathcal{N}}$$

($\hat{\mathcal{N}}$ does not have inverse)

$$\hat{\mathcal{N}}^2 = 1 + \hat{\eta}$$

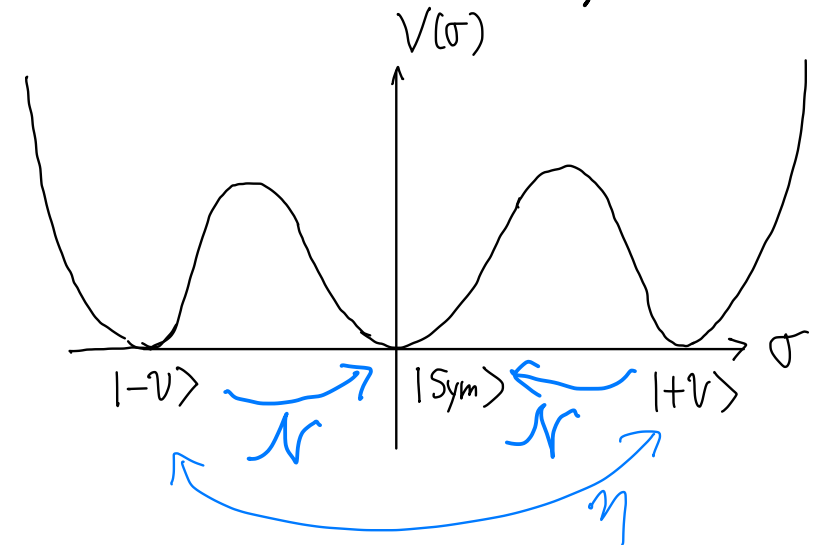
all commute with $\hat{H}$

⇓

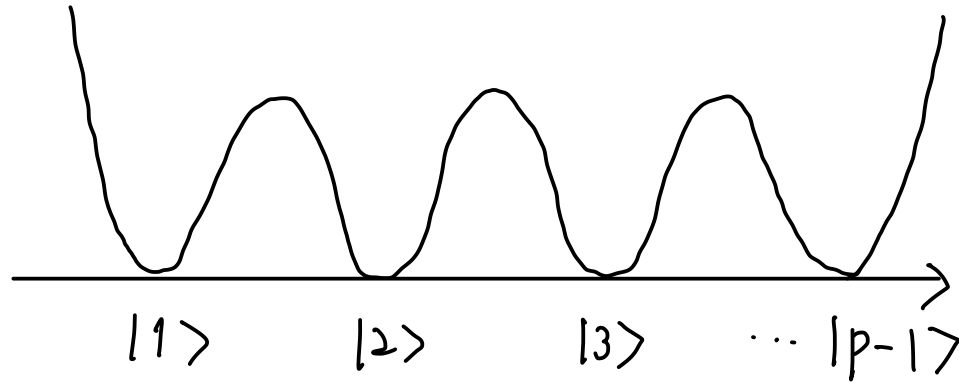
In a gapped phase, at least 3 vacua

$|+\nu\rangle, |-\nu\rangle, |\text{Sym}\rangle$

$$\hat{\mathcal{N}} |+\nu\rangle = \hat{\mathcal{N}} |-\nu\rangle = |\text{Sym}\rangle$$



Generalization: SSB of A_p (a fusion category symmetry)



Topological lines

$$L_a$$

$$a=1, \dots, p-1$$

a

Reference vacuum

$$|0\rangle := |1\rangle \Rightarrow |a\rangle = L_a |0\rangle$$

Kink connecting two vacua a, b ($a = b \pm 1$)

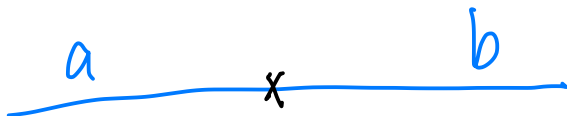
K_{ab} , Same mass due to A_p symmetry.

a | b

Scattering of kinks

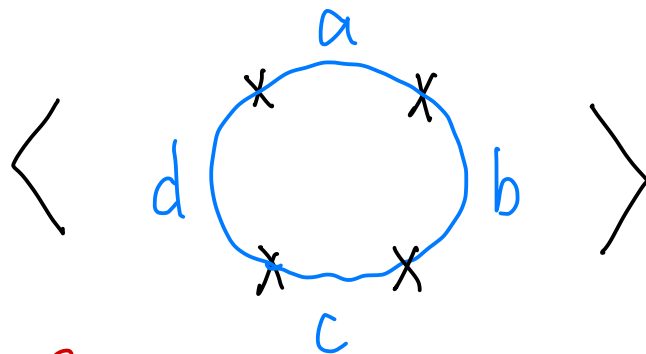
$\phi_{ab}(x)$: Non-genuine local op.

$\parallel$

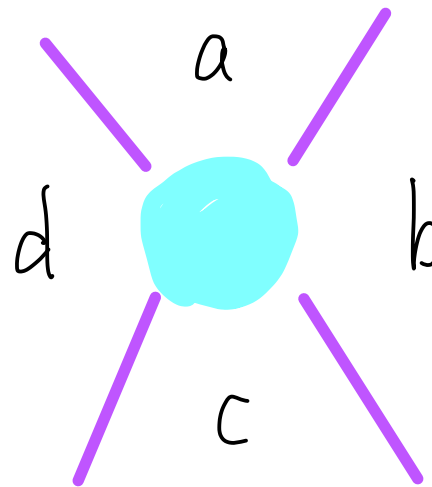


$$\langle K_{ab} | \phi_{ab}(x) | 0 \rangle \neq 0$$

$\Rightarrow$



$\uparrow$
expectation value in $|0\rangle$



$$\Rightarrow S_{dc}^{ab}(\theta)$$

Relative rapidity
Lorentz invariant kinematical parameter

will be related to S_{dc}^{ab} by “LSZ formula”

How to **amputate** ?

Amputate

Conventional LSZ formula,
genuine local operators

$$\left\langle \begin{array}{cc} 1 & x \\ & x 2 \end{array} \right\rangle = \left\langle \begin{array}{c} 1 x \\ 1 x \end{array} \right\rangle \left\langle \begin{array}{c} x 2 \\ x 2 \end{array} \right\rangle$$

1 2 in

Here, non-genuine local operators

$$\left\langle \begin{array}{cc} d & c \\ & c b \end{array} \right\rangle \neq \left\langle \begin{array}{c} d c \\ d c \end{array} \right\rangle \left\langle \begin{array}{c} c b \\ c b \end{array} \right\rangle \quad \text{in}$$

**This should be
used to amputate!**

Amputate

Relation is determined only by the symmetry.

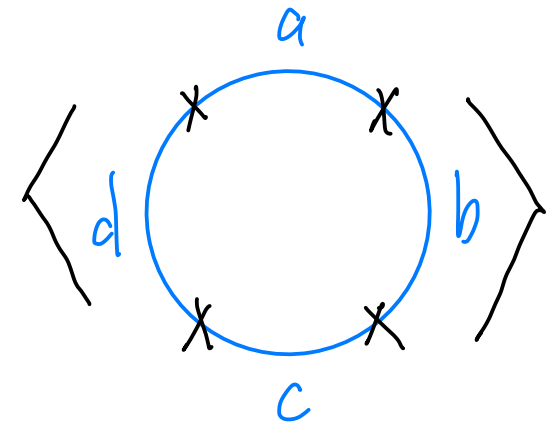
$$\left\langle \text{diagram with 4 external legs } d, c, b, c \right\rangle = \frac{1}{d_c} \left\langle \text{diagram with 2 external legs } d, c \right\rangle \left\langle \text{diagram with 2 external legs } c, b \right\rangle$$

“quantum dimension”: determined only by symmetry (independent of dynamics).

Modified LSZ formula

$$S_{d\ c}^{a\ b}(\theta) = \sqrt{d_c\ d_a} \text{ (Amputate by 2pt functions)}$$

↑
modify the crossing symmetry



Modified crossing relation

$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

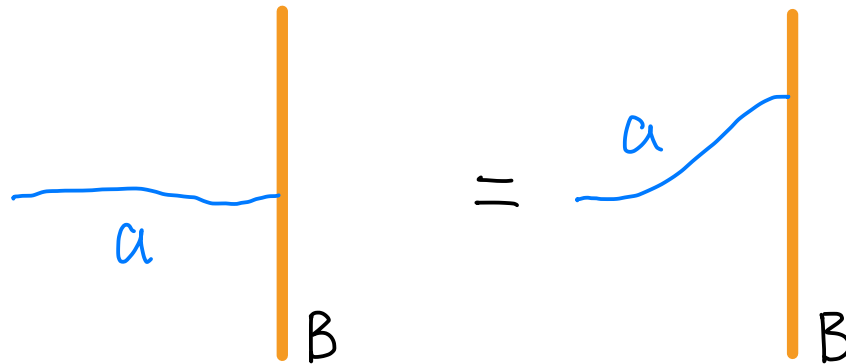
First obtained by [Copetti, Cordova, Komatsu 24]

Boundary scattering

Setup

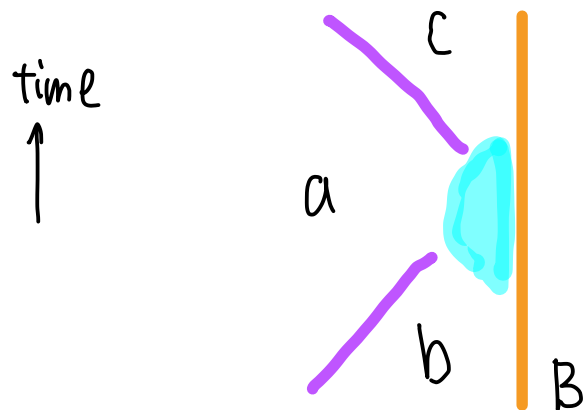
1+1 dim gapped QFT with A_p symmetry

With **weakly symmetric** boundary B



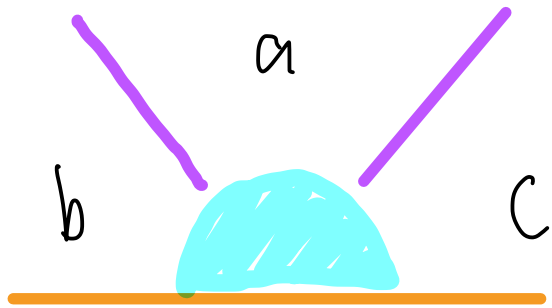
Scattering of a kink

Initiated by [Ghoshal, Zamolodchikov 94]



$$|K_{ab}, \theta, B\rangle_{in} = R_{bc}^a(\theta) |K_{ac}, \theta, B\rangle_{out} + \dots$$

Boundary and particles



$|B\rangle$ · boundary state

$$|B\rangle = \hat{R}_{bc}^a \left(\frac{i\pi}{2} - \theta \right) |K_{ba}, \theta, K_{ac}, \theta\rangle_{out} + \dots$$

$$\hat{R}_{bc}^a \quad \text{vs} \quad R_{bc}^a$$

~

Precise relation?

Exchange time direction

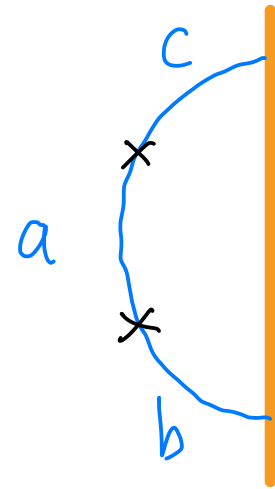
both $\hat{R}_{bc}^a$ R_{bc}^a $\sim$ correlation function

[Ghoshal, Zamolodchikov 94]

(When the symmetry is invertible)

$$\hat{R}_{bc}^a = C_B R_{bc}^a$$

↑ independent of symmetry



[Shimamori, SY]

$$\hat{R}_{bc}^a = \sqrt{d_a F_b^B F_c^B} R_{bc}^a$$

only by symmetry

↑ may depends on the details of the theory

⇐ Careful consideration of the amputation

[Ghoshal, Zamolodchikov 94]

Unitarity
Bulk integrability

[Shimamori, SY]

$$\Rightarrow \hat{R}_{ca}^b\left(\frac{i\pi}{2} - \theta\right) = \sum_d S_{cd}^{ba}(2\theta) \hat{R}_{ca}^d\left(\frac{i\pi}{2} + \theta\right)$$

[GZ] $\Downarrow$

$$R_{ca}^b\left(\frac{i\pi}{2} - \theta\right) = \sum_d S_{cd}^{ba}(2\theta) R_{ca}^d\left(\frac{i\pi}{2} + \theta\right)$$

"Boundary crossing relation"

$$R_{ca}^b\left(\frac{i\pi}{2} - \theta\right) = \sum_d \sqrt{\frac{d_d}{d_b}} S_{cd}^{ba}(2\theta) R_{ca}^d\left(\frac{i\pi}{2} + \theta\right)$$

Comments:

- WT identity, bulk integrability, modified boundary crossing relation lead to a non-trivial consistency condition for S-matrix. [Copetti, Cordova, Komatsu]'s modified solution satisfies this condition.
- Combined with boundary integrability, the solution for $R_{bc}^a(\theta)$ is obtained.

Summary and discussion

Conclusion:

Non-invertible symmetry is important also in scattering problems

- SSB of non-invertible symmetry. Scattering of kinks.

Careful treatment is needed in the derivation of the LSZ formula and its boundary version.

$\Rightarrow$ Non-trivial modification of known formula

Future prospects:

- Bootstrap approach using modified (boundary) crossing relation. (Eg. [Copetti, Cordova, Komatsu 2408])
- Higher dimensions.

Failure of the “assumption” is rather common.

Eg. QED $\langle \text{electron} | \psi(x) | 0 \rangle = 0$
 $\forall \psi(x), \text{local, gauge invariant}$
 $\sim$ IR problems ?

- Callan-Rubakov problem of monopole-electron scattering

$\Rightarrow$ S wave $\rightarrow$ 2 dim scattering particle with a boundary

[van Beest, Boyle Smith, Delmastro, Komargodski 23]