On the index of Dirac operator on the lattice

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Based on

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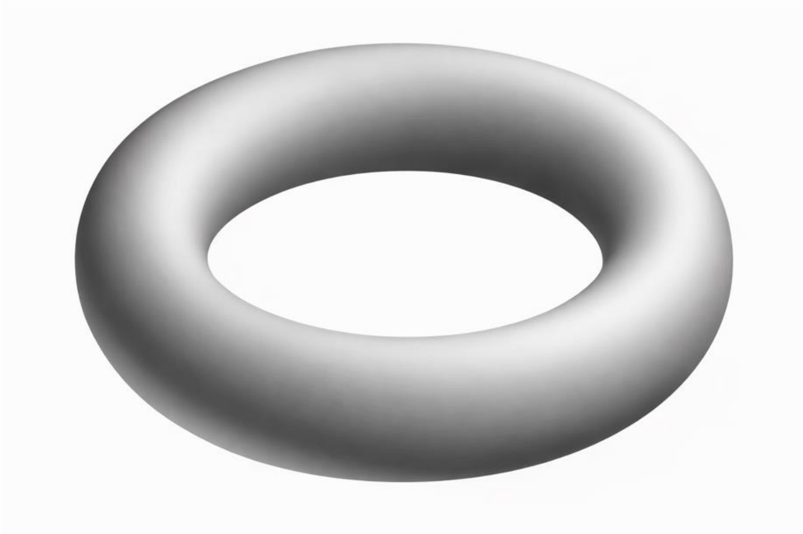
Shinichiroh Matsuo, Tetsuya Onogi, SY
2407.17708, 2503.23921, 2602.12576

Main result

Fredholm index of Dirac type operator on a smooth manifold.



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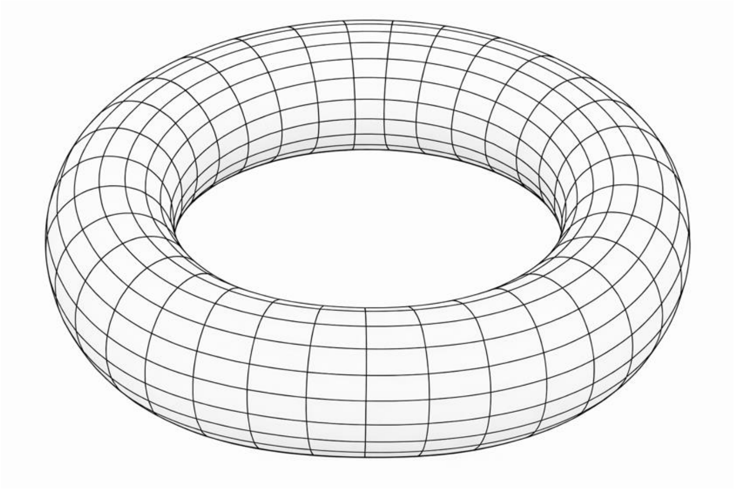
"Index" on a sufficiently fine lattice



Plan

- Why lattice?
- What is the problem with the index on the lattice?
- Formulation

Why Lattice?

Because we need it in the formulation of

quantum field theory.

Finite dimensional integral

Integral (probability theory) (statistical mechanics)

Eg. $\mathcal{H}$: **finite-dimensional** Hermitian vector space.

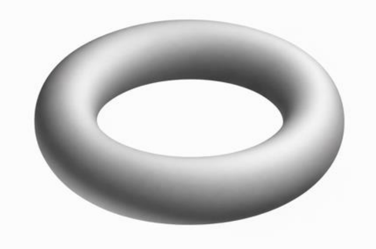
$S : \mathcal{H} \to \mathbb{C}$ continuous

$$Z = \int_{\mathcal{H}} D\,\phi \exp\bigl(-S(\phi)\bigr)$$ is rigorously defined!

Infinite dimensional integral ?

Physicists wants to consider something like

$\mathcal{H}$: **infinite-dimensional** Hilbert space



Eg. X : Remannian manifold

$E \to X$: Hermitian vector bundle

$\mathcal{H} := \Gamma(E)$

??

$$Z = \int_{\mathcal{H}} D\,\phi \exp\left(-S(\phi)\right)$$

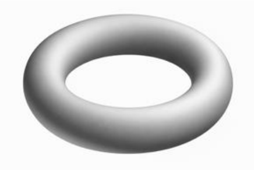
"path-integral", "functional integral"

Idea

Approximate X by finite number of points

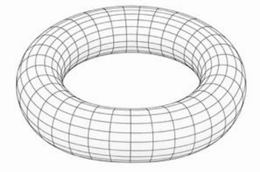
Eg.

$X = \mathbb{R}^n/\mathbb{Z}^n$



$a = 1/N, N \in \mathbb{Z}$

$X_a = (a\mathbb{Z})^n/\mathbb{Z}^n$



Lattice!

$E_a := E\big|_{X_a}$

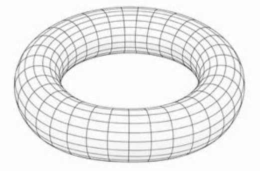
$\mathcal{H}_a = \Gamma(E_a)$ **is finite-dimensional**

Lattice

$a = 1/N, N \in \mathbb{Z}$

$X_a = (a\mathbb{Z})^n/\mathbb{Z}^n$

$E_a := E\big|_{X_a}$



$\mathcal{H}_a = \Gamma(E_a)$ is finite-dimensional

Find "nice" $S_a : \mathcal{H}_a \to \mathbb{C}$

$$Z_a := \int_{\mathcal{H}_a} D\phi \exp\left(-S_a(\phi)\right)$$ **is rigorously defined**

"regularization"

continuum limit

$Z := \lim_{a \to 0} Z_a$: continuous QFT

So far, the best way to define QFT nonperturbatively.

Why lattice?

Because we need it in the formulation of

quantum field theory.

Index on the lattice

What is the problem?

What is the problem?

Naively, index on the lattice is always 0.

So, it cannot be formulated straightforwardly.

Set up

X : closed Remannian manifold

$E \to X$: Hermitian vector bundle
with $\mathbb{Z}_2$ grading

$\gamma: E \to E,\qquad \gamma^2 = 1 \quad \Rightarrow \quad E = E_+ \oplus E_-$
eigenspace of $\gamma = \pm 1$

Assume $\mathrm{rank}\ E_+ = \mathrm{rank}\ E_-$

$D: C^\infty(X;E) \to C^\infty(X;E)$

formally self-adjoint, first order, elliptic differential operator

$$\gamma D + D\gamma = 0 \quad \text{“odd”}$$

Eg. Dirac operator in even dimensions.

$$\gamma^2 = 1, \qquad D\gamma + \gamma D = 0$$

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Index:

$$\mathrm{Ind}(D) = \dim \mathrm{Ker} D_+ - \dim \mathrm{Ker} D_-$$

Go to lattice!

Suppose straightforward discretization exists,

Everything becomes finite-dimensional,

$$\mathcal{H}_a = \mathcal{H}_{a+} \oplus \mathcal{H}_{a-} \qquad \dim \mathcal{H}_{a+} = \dim \mathcal{H}_{a-} = (\operatorname{rank} E_+) \times |X_a|$$

$D_a : \mathcal{H}_a \to \mathcal{H}_a$ self-adjoint $\quad \gamma_a D_a + D_a \gamma_a = 0$

$$\Rightarrow \quad D_a = \begin{pmatrix} 0 & D_{a_-} \\ D_{a_+} & 0 \end{pmatrix}$$

$$\Rightarrow \quad \operatorname{Ind}(D_a) \coloneqq \dim \operatorname{Ker} D_{a+} - \dim \operatorname{Ker} D_{a-} = \dim \mathcal{H}_{a+} - \dim \mathcal{H}_{a-} = 0$$

What is the problem?

Naively, index on the lattice is always 0.

So, it cannot be formulated straightforwardly.

Such D_a cannot approximate D

due to so-called “doublers”

History

[Wilson 75, 77] Noticed "doubler problem." He added "Wilson term" to solve this problem.

His lattice Dirac operator is called "Wilson Dirac operator"

$$D_W$$

It is **not odd** $\gamma D_W + D_W \gamma \neq 0$

Index is not defined and no contradiction to our observation

[Nielsen, Ninomiya 81] No-go theorem

One cannot discretize Dirac operator while keeping oddness.

Is it impossible to formulate index on the lattice?

[Itoh, Iwasaki, Yoshie 87] Index on the lattice by the **spectral flow.**

[Neuberger 98], [Hasenfratz, Laliena, Niedermayer 98]…

Index from "overlap Dirac operator" that satisfy Ginsparg-Wilson relation

$$\gamma D_{\mathrm{ov}} + D_{\mathrm{ov}}\,\gamma = a\, D_{\mathrm{ov}} \gamma D_{\mathrm{ov}}$$

[Ginsparg, Wilson 82]

Formulation

Idea

Realizing $\gamma D + D\gamma = 0$ on the lattice is impossible

Fact: suspension isomorphism of K-theory:

Considering $(\mathcal{H}, D, \gamma)$ is equivalent to one parameter family $(\mathcal{H}, D + t\,\gamma), -1 \leq t \leq 1$ "spectral flow"

We do not need $\gamma D + D\gamma = 0$ in the formulation.

$\Downarrow$

Formulated on the lattice.

K-theory

(For rigorous definition for our purpose, see our latest paper 2602.12576)

$$K^0(pt) := \{(\mathcal{H}, h, \gamma)\}/\sim$$

$\mathcal{H}$: Hilbert space $\quad h, \gamma: \mathcal{H} \to \mathcal{H}$, self-adjoint $\quad \gamma^2 = 1, h\gamma + \gamma h = 0$

$(\mathcal{H}, h, \gamma) \sim (\mathcal{H}', h', \gamma')$ if related by a continuous deformation.

$(\mathcal{H}, h, \gamma) \sim (\mathcal{H} \oplus \mathcal{H}', h \oplus h', \gamma \oplus \gamma')$ if h' is invertible.

Abelian group by

$$[(\mathcal{H}, h, \gamma)] + [(\mathcal{H}', h', \gamma')] := [(\mathcal{H} \oplus \mathcal{H}', h \oplus h', \gamma \oplus \gamma')]$$

Fact: $K^0(pt) \cong \mathbb{Z}$ as abelian group

Index: $$\mathrm{Ind}(D) = [(\mathcal{H}, D, \gamma)] \in K^0(pt)$$

K-theory

(For rigorous definition for our purpose, see our latest paper 2602.12576)

$$K^{-1}(I,\, \partial I) \coloneqq \{(\mathcal{H}\_\mathrm{t}, h\_\mathrm{t})\}/\sim \qquad I = [-1,1], \partial I = \{-1, +1\}$$

$t \in I$ $\quad$ $\mathcal{H}_t$: continuous family of Hilbert space

$h_t : \mathcal{H}_t \to \mathcal{H}_t$, continuous family of self-adjoint operators

For $t \in \partial I$, h_t is invertible.

$\sim$: similar as before

Abelian group by $\oplus$

Fact: suspension isomorphism

$$K^0(pt) \cong K^{-1}(I,\, \partial I)$$

$$[(\mathcal{H}, h, \gamma)] \mapsto [(\mathcal{H}, h + t\gamma)]$$

(inverse map is not simple)

Fact: $K^0(pt) \cong K^{-1}(I, \partial I)$

$$[(\mathcal{H}, h, \gamma)] \mapsto [(\mathcal{H}, h + t\gamma)]$$

$$\mathrm{Ind}(D) = [(\mathcal{H}, D, \gamma)] \mapsto [(\mathcal{H}, D + t\gamma)] \in K^{-1}(I, \partial I)$$

We can argue index here!

We do not need $\gamma D + D\gamma = 0$

Theorem

$X = \mathbb{R}^n/\mathbb{Z}^n$

$E \to X$: Hermitian vector bundle

$\gamma: E \to E, \qquad \gamma^2 = 1$

$D: C^\infty(X;E) \to C^\infty(X;E), \qquad \gamma D + D\gamma = 0$

$a = 1/N, N \in \mathbb{Z} \qquad X_a = (a\mathbb{Z})^n/\mathbb{Z}^n \subset X \qquad E_a := E\big|_{X_a}$

$\Downarrow$

$D_W: E_a \to E_a$ is defined (Wilson Dirac operator).

For sufficiently small a (large N)

$$[(\mathcal{H}, D + t\gamma)] = [(\mathcal{H}_a, D_W + t\gamma)] \in K^{-1}(I, \partial I)$$

Atiyah-Patodi-Singer index

Index for manifold with boundary.

We can also formulate APS index in our formulation by making use of domain-wall fermion description of APS index.

[Fukaya, Onogi, SY 17], [Fukaya, Furuta, Matsuo, Onogi, Yamashita, SY 19, 20]

This is the first mathematically rigorous formulation of APS index on the lattice.

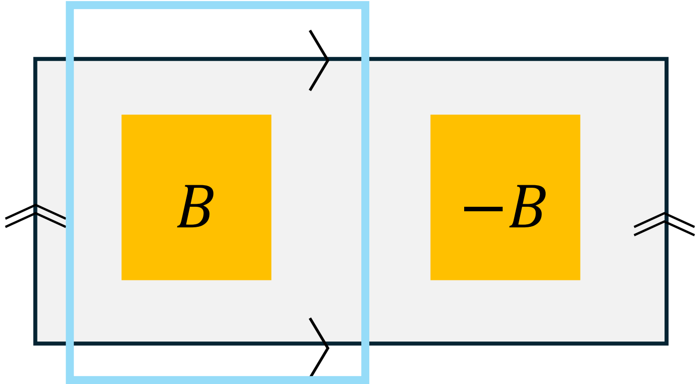
Numerical calculation

One can numerically calculate index once it is put on the lattice.

$X = T^2$

U(1) gauge theory

(Spin^c bundle with connection)



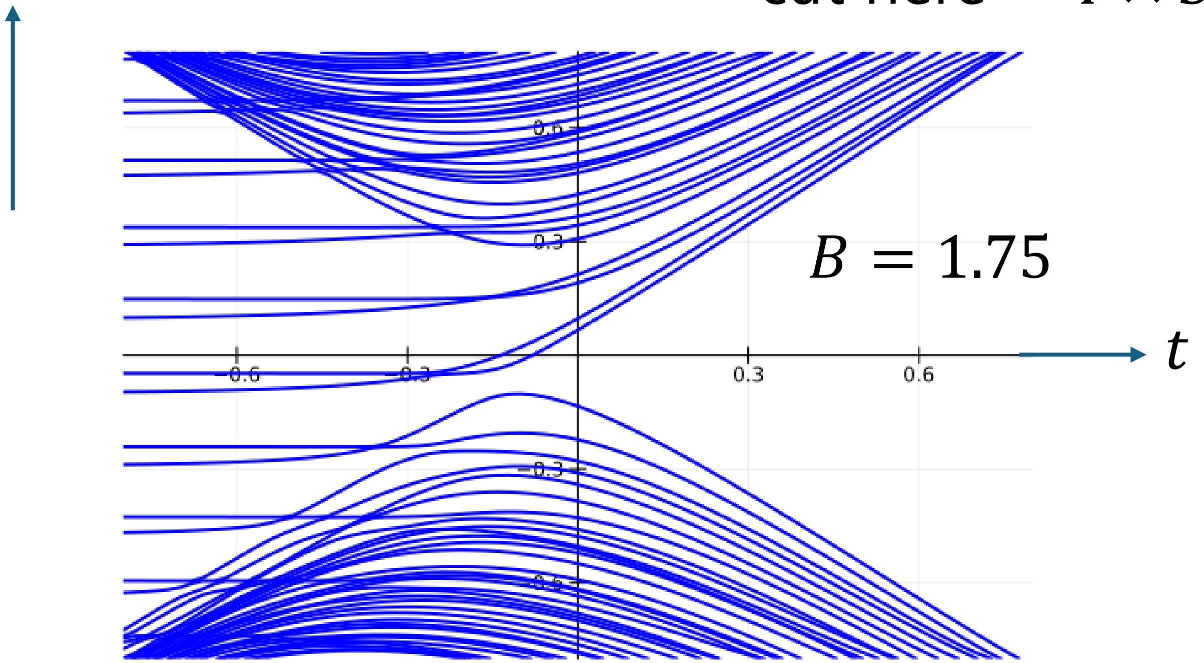
cut here → $I \times S^1$

Put magnetic flux B and $-B$

total flux = (first Chern class) = 0

Known value of APS index = round B

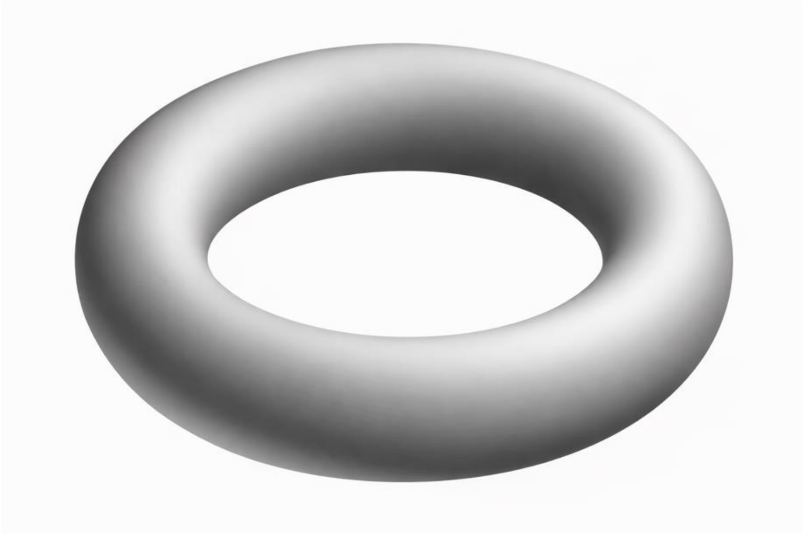
Eigenvalues of domain-wall Dirac operator



Two zero-crossings ⇨ Ind = 2

Main result

Fredholm index of Dirac type operator on a smooth manifold.



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