素粒子物理学特論 I

Theoretical Particle Physics I

Path Integral Methods in Quantum Field Theory

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0. Introduction

☆ Quantum Field Theory (QFT)

~ A kind of Quantum Mechanics (QM)

degrees of freedom at each point on the space

~ field

- Describe the physics of elementary particles.

☆ Formulation of QM

① Operator formalism

Hilbert space, Hamiltonian

② Path integral formalism

~ "Very large dimensional integral"
(infinite)

- Both of them are useful

① is useful as you know

② is also useful as I will explain in this lecture

You should learn both of them and relation between them, and utilize them.

☆ Is path integral useful ?

In path integral

- Time and space are equaly treated.

  ※ In operator formalism, this is not the case

- Easy to derive Feynman rule.

  In particular, when including derivative interactions, such as non-abelian gauge theory.

- Euclidean path integral formalism

  - Renormalization group

  - Saddle point approximation

☆ Difficulty in learning QFT

Physics

Physics

Abstract "Theory"	Fields, canonical quantization, effective action, action, path integral, Feynman rule, ...	Scattering cross section, Energy spectrum

Intermediate steps, not physics
Very long !

compared to undergraduate subjects like
classical mechanics, electromagnetism,
quantum mechanics, ...

- Be patient.

- Be aware where you are.

1. Review of Quantum Mechanics

☆ States and observables

- Hilbert space $\mathcal{H}$: Complex vector space with Hermitian inner product

  finite or infinite dimensions

- State $|\psi\rangle \in \mathcal{H} \setminus \{0\}$

  $c \in \mathbb{C} \setminus \{0\}$ $\quad |\psi\rangle \sim c|\psi\rangle$ "same state"

- Observable $\hat{A}$: Hermitian op. on $\mathcal{H}$

  Expectation value

$$\langle \hat{A} \rangle_\psi = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}$$

$\hat{A}$ is diagonalizable

maybe integral

$$\hat{A}|a\rangle = a|a\rangle \quad (a \in \mathbb{R}), \quad \sum_a |a\rangle\langle a| = 1$$

$$\sum_a |a\rangle\langle a| \, a = \hat{A}$$

☆ Time evolution

- Hamiltonian $\hat{H}$: (an observable) (Energy)

- $|\psi(t)\rangle$: state at time t

  Satisfy
  $$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$
  "Schrödinger equation"

  When $\hat{H}$ is independent of t, $\exists$ formal solution
  $$|\psi(t)\rangle = e^{-it\hat{H}}\underbrace{|\psi(0)\rangle}_{=:|\psi\rangle}$$

- Expectation value at time t
  $$\langle\hat{A}\rangle_\psi(t) = \frac{\langle\psi(t)|\hat{A}|\psi(t)\rangle}{\langle\psi|\psi\rangle}$$

"Schrödinger picture"

- States depend on time
- Observables do not implicitly depends on time

  (Can depend explicitly time)

☆ Heisenberg picture

$$\langle \hat{A} \rangle_\psi (t) = \frac{\langle \psi(t) | \hat{A} | \psi(t) \rangle}{\langle \psi | \psi \rangle}$$

$$= \frac{\langle \psi | \hat{A}(t) | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$\hat{A}(t) := e^{it\hat{H}} \hat{A} e^{-it\hat{H}}$$

"Heisenberg operator"

Satisfy

$$\frac{d}{dt} \hat{A}(t) = i [\hat{H}, \hat{A}(t)]$$

"Heisenberg equation"

- State is independent of time.
- Observable depends on time.

※ In the operator formalism of QFT, Heisenberg picture is by far more often used than Schrödinger picture

☆ Examples

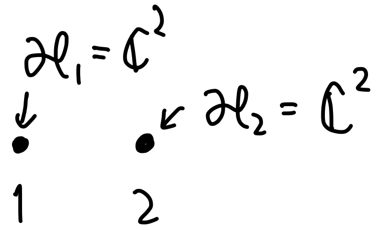
- $\mathcal{H} = \mathbb{C}^2$ "single spin"

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \quad \psi_0, \psi_1 \in \mathbb{C}$$

$\hat{H}$: 2×2 Hermitian matrix

Eg: $\hat{H} = -\varepsilon \sigma^1 \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- Two spins



Combine two systems $\Rightarrow$ Total Hilbert space is the tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Observables

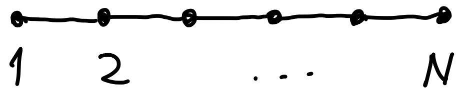
$\sigma_1^i = \sigma^i \otimes 1 \qquad \sigma^i$ acting on $\mathcal{H}_1$

$\sigma_2^i = 1 \otimes \sigma^i \qquad$ " $\qquad \mathcal{H}_2$

Eg: $\hat{H} = J\, \sigma_1^1\, \sigma_2^1$

Spin chain

put a spin on each site



$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \qquad \mathcal{H}_n = \mathbb{C}^2$$

observable

$$\sigma^i_n := 1 \otimes \cdots \otimes 1 \otimes \underset{\substack{\uparrow \\ n}}{\sigma^i} \otimes 1 \otimes \cdots \otimes 1$$

σ^i acting on site n

Hamiltonian

$$\hat{H} = \sum_{i=1,2,3} \sum_{n=1}^{N-1} J_i \, \sigma^i_n \, \sigma^i_{n+1}$$

※ You may obtain QFT by $\boxed{\text{Nontrivial}}$ appropreate continuum limit
(dof at each pt in space) $\begin{pmatrix} \text{lattice spacing} \to 0 \\ N \to \infty \end{pmatrix}$

of spin chain

$\mathcal{H} = L^2(\mathbb{R})$ (QM of a particle on a line)

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{\infty} dq \, |f(q)|^2 < +\infty \right\} / \sim$$

$$f \sim g \overset{\text{def}}{\Longleftrightarrow} \int_{-\infty}^{\infty} dq \, |f(q) - g(q)|^2 = 0$$

$$\hat{H} = -\frac{1}{2}\left(\frac{d}{dq}\right)^2 + V(q)$$

"potential"

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \quad (\cong L^2(\mathbb{R}^N))$$

$$\mathcal{H}_n = L^2(\mathbb{R})$$

coordinate q_n

Eg.

$$\hat{H} = -\frac{1}{2}\sum_n \left(\frac{\partial}{\partial q_n}\right)^n + V(q)$$

$$V(q) = \sum_{n=1}^{N-1} \frac{1}{2}k(q_{n+1} - q_n)^2 + \sum_{n=1}^{N}\left(\frac{1}{2}\ell q_n^2 + \frac{\lambda}{4!} q_n^4\right)$$

$\Leftrightarrow$ 

※ You may also obtain QFT from this model by taking appropreate continuum limit

nontrivial

☆ Density matrix

Eg. a spin, $\mathcal{H} = \mathbb{C}^2$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Consider a state prepared by classical probability (Eg. dice, coin toss)

"mixed state" ($\Leftrightarrow$ "pure state") prob. $p_0 \to |0\rangle$ $\qquad p_0 + p_1 = 1$

prob. $p_1 \to |1\rangle$ $\qquad p_0, p_1 \geq 0$

Expectation value

$$\langle \hat{A} \rangle = p_0 \langle 0|\hat{A}|0\rangle + p_1 \langle 1|\hat{A}|1\rangle$$

$$= \mathrm{Tr}(\hat{\rho}\hat{A})$$

$$\hat{\rho} := p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1| \qquad \text{"density matrix"}$$

In general,

$\hat{\rho}$: Hermitian, all eigenvalues ≥ 0, $\mathrm{Tr}\,\hat{\rho} = 1$

can represent a (mixed or pure) state

Expectation value

$$\langle \hat{A} \rangle_{\hat{\rho}} = \mathrm{Tr}(\hat{\rho}\hat{A})$$

□ Examples

- pure state $|\psi\rangle \Longrightarrow \hat{\rho}_\psi := \dfrac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$

- Canonical ensemble (thermal ensemble)

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}, \quad Z := \mathrm{Tr}\, e^{-\beta \hat{H}} \qquad \text{"partition function"}$$

$\beta > 0$ inverse temperture.

Components of $e^{-\beta\hat{H}}$ includes useful information

$|\phi\rangle$: basis , $|n\rangle$: Energy basis $\hat{H}|n\rangle = E_n|n\rangle$

$$\langle\phi'|e^{-\beta\hat{H}}|\phi\rangle = \sum_n \langle\phi'|n\rangle\langle n|e^{-\beta\hat{H}}|\phi\rangle$$

$$= \sum_n e^{-\beta E_n} \langle\phi'|n\rangle\langle n|\phi\rangle$$

$$= e^{-\beta E_0}\langle\phi'|0\rangle\langle 0|\phi\rangle + e^{-\beta E_1}\langle\phi'|1\rangle\langle 1|\phi\rangle + \cdots$$

$\Downarrow$

E_n, Energy eigenvalues

$\langle\phi'|n\rangle\langle n|\phi\rangle$: density matrix element of $|n\rangle$

$\sim$ wave function $\langle\phi|n\rangle$

2. From Operator to Path Integral

Sketch

Operator formalism $\xrightarrow{\text{now}}$ Path integral formalism

$\mathcal{H}$, $\hat{H}$ $\xleftarrow{\text{later}}$ Classical statistical mechanics

equivalence

$$Z = \sum_{\substack{\phi \\ \text{all conf.}}} e^{-S_E(\phi)}$$

☆ Generic strategy

$\phi \in \Lambda$: label of basis

$|\phi\rangle$: basis of $\mathcal{H}$

(Eg. $\Lambda = \{0, 1\}$ for spin
$\Lambda = \mathbb{R}$ for a particle on a line)

$$\sum_{\phi} |\phi\rangle\langle\phi| = 1$$

(may be integral
Eg. $\int d\phi \, |\phi\rangle\langle\phi| = 1$)

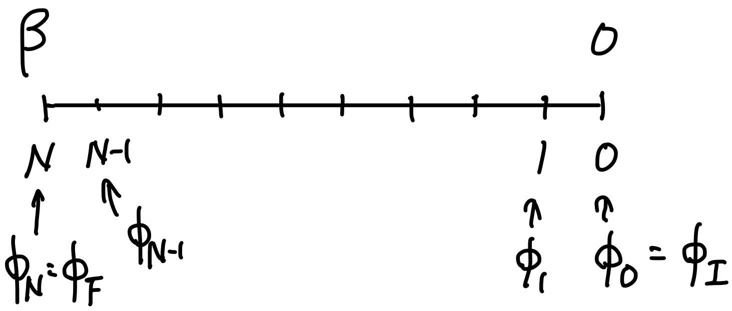
□ Consider

$$A := \langle \phi_F | e^{-\beta \hat{H}} | \phi_I \rangle \qquad \phi_F, \phi_I \in \Lambda$$

$$= \langle \phi_F | e^{-a\hat{H}} e^{-a\hat{H}} \cdots e^{-a\hat{H}} | \phi_I \rangle$$

$$(aN = \beta)$$

$$= \underbrace{\sum_{\phi_{N-1}, \phi_{N-2}, \cdots, \phi_1}}_{=: \sum_{\{\phi\}}} \langle \phi_F | e^{-a\hat{H}} | \phi_{N-1} \rangle \langle \phi_{N-1} | \cdots \cdots \langle \phi_1 | e^{-a\hat{H}} | \phi_0 \rangle$$



$$= \sum_{\{\phi\}} \prod_{n=0}^{N-1} \langle \phi_{n+1} | e^{-a\hat{H}} | \phi_n \rangle$$

Define $\tilde{L}_{\phi'\phi}$ by

$$\langle \phi' | e^{-a\hat{H}} | \phi \rangle = e^{-a\tilde{L}_{\phi'\phi}}$$

$$\tilde{S}_{E,a}(\phi) := \sum_{n=0}^{N-1} a \tilde{L}_{\phi_{n+1}\phi_n}$$

$$\Rightarrow \quad A = \sum_{\substack{\{\phi\} \\ \phi_N = \phi_F,\ \phi_0 = \phi_I}} e^{-\tilde{S}_{E,a}(\phi)}$$

□ Problem : $\tilde{S}_{E,a}(\phi)$ is not simple enough for given simple $\hat{H}$

$\Downarrow$

Find simple $S_{E,a}(\phi)$ such that

$$\begin{cases} S_{E,a}(\phi) \approx \tilde{S}_{E,a}(\phi) \quad \text{for small } a \\ A_a := \sum_{\{\phi\}} e^{-S_{E,a}(\phi)} \\ A = \lim_{a\to 0} A_a \end{cases}$$

finite a : "cut off theory"
"regularized theory"

□ Eg: Spin $\phi \in \{+1, -1\}$

$$\hat{H} = -\varepsilon\sigma_1$$

$$e^{-a\hat{H}} \approx 1 - a\hat{H} = \begin{pmatrix} 1 & a\varepsilon \\ a\varepsilon & 1 \end{pmatrix}$$

$$a L_{\phi'\phi} = \begin{cases} 0 & \text{if } \phi' = \phi \\ -\log(a\varepsilon) & \text{if } \phi' \neq \phi \end{cases}$$

$$= +J_a(1 - \phi'\phi)$$

$$J_a = -\frac{1}{2}\log(a\varepsilon) > 0 \quad \text{for small } a$$

$$S_{E,a}(\phi) = \sum_n J_a (1 - \phi_{n+1}\phi_n)$$

$$= -J_a \sum_n \phi_{n+1}\phi_n + \underbrace{\sum_n J_a}_{\phi \text{ independent, local}}$$

$$A_a = \sum_{\{\phi\}} \exp\left(J_a \sum_n \phi_{n+1}\phi_n + \sum_n J_a \right)$$

$\Rightarrow$ 1-dim classical Ising model

☐ Eg.

$$\phi \in \mathbb{R}$$

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{\phi}) \qquad \left([\hat{\phi}, \hat{p}] = i \right)$$

$$\langle \phi' | e^{-a\hat{H}} | \phi \rangle$$

$$\approx \langle \phi' | e^{-a\frac{1}{2m}\hat{p}^2} e^{-aV(\hat{\phi})} | \phi \rangle$$

$$\left(e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\cdots} \right)$$

$$= \underbrace{\langle \phi' | e^{-a\frac{1}{2m}\hat{p}^2} | \phi \rangle}\, e^{-aV(\phi)}$$

$$\underline{\qquad} = \int dp\, \underbrace{\langle \phi' | p \rangle}_{\frac{1}{\sqrt{2\pi}} e^{ip\phi'}} \underbrace{\langle p | e^{-a\frac{1}{2m}\hat{p}^2} | \phi \rangle}_{= e^{-a\frac{1}{2m}p^2} \frac{1}{\sqrt{2\pi}} e^{-ip\phi}}$$

$$= \int \frac{dp}{2\pi}\, e^{\underbrace{ip\overbrace{(\phi'-\phi)}^{a\dot{\phi}} - \frac{a}{2m}p^2}_{=}}$$

complete square

$$-\frac{a}{2m}\left(p^2 - 2ip\dot{\phi}m\right) \qquad \left(\dot{\phi} := \frac{\phi'-\phi}{a} \text{ short hand}\right)$$

$$= -\frac{a}{2m}\left((p - im\dot{\phi})^2 + m^2\dot{\phi}^2\right)$$

$$= -\frac{a}{2m}\left(p - i\frac{m}{a}\dot{\phi}\right)^2 - a\frac{1}{2}m\dot{\phi}^2$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{a}}\, e^{-a\frac{1}{2}m\dot{\phi}^2}$$

$$= \sqrt{\frac{m}{2\pi a}}\, e^{-a\frac{1}{2}m\dot{\phi}^2}$$

$$\langle \phi' | e^{-a\hat{H}} | \phi \rangle \approx \sqrt{\frac{m}{2\pi a}}\, e^{-a\left(\frac{1}{2}m\left(\frac{\phi'-\phi}{a}\right)^2 + V(\phi)\right)}$$

$$\int D\phi := \int \prod_{n=1}^{N-1} d\phi_n \times \left(\sqrt{\frac{m}{2\pi a}}\right)^N$$

$$S_{E,a}(\phi) = \sum_{n=0}^{N-1} a\left(\frac{1}{2}m\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2 + V(\phi_n)\right)$$

$$A_a = \int D\phi \, e^{-S_{E,a}(\phi)}$$

$$\left(\begin{array}{l} \phi_0 = \phi_I \\ \phi_N = \phi_F \end{array}\right)$$

Naive limit

Above example

Introduce $\phi(\tau)$ $\quad 0 \leq \tau \leq \beta$

s.t. $\phi(an) = \phi_n$

$$\frac{\phi_{n+1} - \phi_n}{a} \approx \frac{d\phi}{d\tau}(\tau = an)$$

$$\sum_{n=0}^{N-1} a \approx \int_0^\beta d\tau$$

If a is very small

$$S_E(\phi) := \int_0^\beta d\tau \left(\frac{1}{2} m \dot{\phi}^2 + V(\phi) \right)$$

$$S_{E,a}(\phi) \approx S_E(\phi) \quad , \quad \underline{\text{for given } \phi(\tau)}$$

In the literature, we usually write $S_E(\phi)$

instead of $S_{E,a}(\phi)$.

Regularization is implicit.

※ " $\lim_{\substack{a \to 0 \\ (N \to \infty)}} S_{E,a}(\phi) = S_E(\phi)$ " make sense ?

$S_{E,a}(\phi_1, \phi_2, \cdots, \phi_{N-1})$

$S_{E,a} : \mathbb{R}^{N-1} \to \mathbb{R}$

How you can compare two such functions with different domain of definition ?

$\lim_{a \to 0} A_a = A$ make sense

You have to take $a \to 0$ limit **AFTER** evaluating the integral.

Typical configulation

$$S_{E,a}(\phi) = \sum_{n=0}^{N-1} \left(\underbrace{\frac{1}{2}\frac{m}{a}(\phi_{n+1}-\phi_n)^2} + a V(\phi_n) \right)$$

v.s.

$$S_E(\phi) = \int d\tau \left(\frac{1}{2} m \dot{\phi}^2 + V(\phi) \right)$$

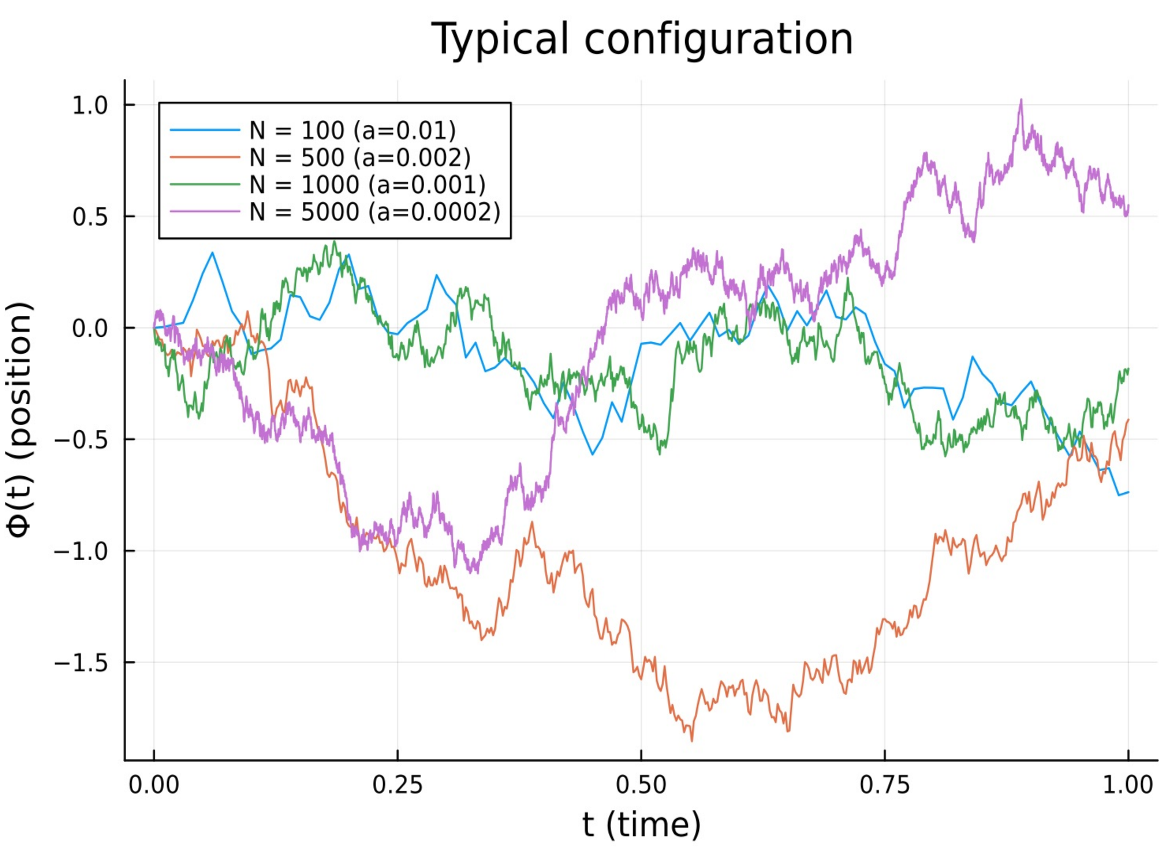
One may say

"Due to this term, the probability of (configulation with large $\frac{\phi_{n+1}-\phi_n}{a}$) is exponentially small.

$\Downarrow$

Smooth configulation dominate for small a"

Is it true?



Why?

Although probability of each configuration is small, number of such configration is huge compared to smooth configuration.

☆ Restore $\hbar$

Use dimensional analysis.

dim of $\hbar$ = dim of S

$$e^{-S_E(\phi)} \Rightarrow e^{-\frac{1}{\hbar}S_E(\phi)}$$

small $\hbar$ $\Rightarrow$ ~ Saddle point approximation. around the minimum of $S_E(\phi)$

※ This is just a nice way of approximation. No physical meaning. (τ is NOT the time)

☆ Lorentzian path integral

T : time interval

$$A = \langle \phi_F | e^{-iT\hat{H}} | \phi_I \rangle \qquad (aN = T)$$

$$= \sum_{\{\phi\}} \prod_{n=0}^{N-1} \langle \phi_{n+1} | e^{-ia\hat{H}} | \phi_n \rangle$$

$$\langle \phi' | e^{-ia\hat{H}} | \phi \rangle = e^{iaL_{\phi'\phi}}$$

$$\tilde{S}_a(\phi) = \sum_{n=0}^{N-1} a L_{\phi_{n+1}\phi_n}$$

Simple

$$S_a(\phi) \approx \tilde{S}_a(\phi) \qquad (a : \text{small})$$

$$A_a = \sum_{\{\phi\}} e^{iS_a(\phi)}$$

<u>Eg.</u> $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{\phi})$

$$\langle \phi' | e^{-ia\hat{H}} | \phi \rangle = \sqrt{\frac{2\pi m}{ia}}\, e^{ia\left(\frac{1}{2}m\dot{\phi}^2 - V(\phi)\right)}$$

(Exercise) $\qquad \left(\dot{\phi} := \frac{\phi' - \phi}{a}\right)$

$$\int D\phi := \int \prod_{n=1}^{N-1} d\phi_n \left(\sqrt{\frac{2\pi m}{ia}}\right)^N$$

$$S_a(\phi) = \sum_{n=0}^{N-1} a\left(\frac{1}{2}m\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2 - V(\phi_n)\right)$$

$$A_a = \int D\phi \, e^{iS_a(\phi)}$$

$$\phi_0 = \phi_I, \ \phi_N = \phi_F$$

For small a

$$\underset{\text{Action}}{S_a(\phi) \approx S(\phi)} = \int_0^T dt\left(\frac{1}{2}m\dot{\phi}^2 - V(\phi)\right)$$

Regularization is implicit.

Restore $\hbar$

$$A_a = \int D\phi \, e^{\frac{i}{\hbar}S_a(\phi)}$$

$\hbar \to 0$ limit

stationary configuration $\dfrac{\partial S_a(\phi)}{\partial \phi_n} = 0$,

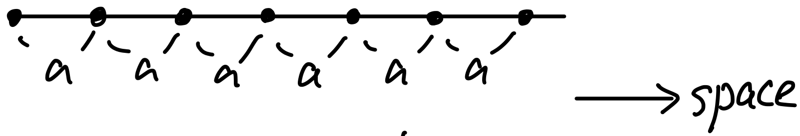
"dominate" classical trajectory

Other configurations cancell to neiboring ones.

☆ QFT

(defined)

QFT $\xleftarrow[\text{continuum limit (subtle)}]{}$ QM with d.o.f. at lattice sites on the space

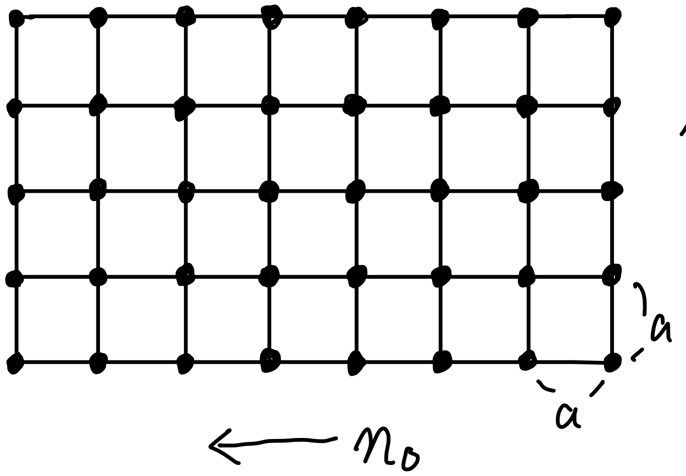


$\Downarrow$ path integral

$$A_a = \int D\phi \, e^{-S_a(\phi)}$$

(cut off theory)

We have to take subtle "continuum limit" <u>in any case</u>.



$\uparrow \vec{n} = (n_1, n_2, \cdots, n_{d-1})$

$\uparrow$ dim of space

Integration (summation) variable at each site on the Euclidean spacetime. $\phi_{n_0, \vec{n}}$

$$\int D\phi \sim \int \prod_{n_0, \vec{n}} d\phi_{n_0, \vec{n}}$$

$$A_a = \int D\phi \, e^{-S_a(\phi)} \Rightarrow \text{d-dim classical statistical mechanics.}$$

□ <u>Eg.</u> 1+1 dim scalar field

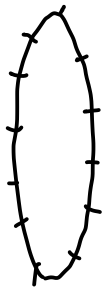
$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{N_1}$$

$$\mathcal{H}_{n_1} = L^2(\mathbb{R})$$

$\phi = (\phi_1, \phi_2, \cdots, \phi_{N_1})$

periodic boundary condition

$\phi_0 = \phi_{N_1}$ for simplicity



$$\hat{H} = \sum_{n_1} \frac{1}{2a} \hat{P}_{n_1}^2 + \sum_{n_1} \frac{1}{2} a \left(\frac{\hat{\phi}_{n_1+1} - \hat{\phi}_{n_1}}{a} \right)^2 + \sum_{n_1=0}^{N_1} a V(\phi_{n_1})$$

Complete basis $|\phi\rangle = |\phi_1, \phi_2, \cdots, \phi_{N_1}\rangle$

$$\hat{\phi}_{n_1} |\phi\rangle = \phi_{n_1} |\phi\rangle$$

$$\int \prod_{n_1} d\phi_{n_1} \, |\phi\rangle\langle\phi| = 1$$

Prepare many labels of complete basis

$$|\phi_{n_0}\rangle = |\phi_{n_0,1}, \phi_{n_0,2}, \cdots, \phi_{n_0,N_1}\rangle$$

$$\hat{\phi}_{n_1} |\phi_{n_0}\rangle = \phi_{n_0,n_1} |\phi_{n_0}\rangle$$

$$\int \prod_{n_1} d\phi_{n_0,n_1} \, |\phi_{n_0}\rangle\langle\phi_{n_0}| = 1$$

$$A = \langle \phi_F | e^{-\beta \hat{H}} | \phi_I \rangle \qquad (\beta = N_0 a)$$

$$= \int \prod_{n_0=1}^{N_0-1} \prod_{n_1} d\phi_{n_0, n_1} \prod_{n_0=0}^{N_0-1} \langle \phi_{n_0+1} | e^{-a\hat{H}} | \phi_{n_0} \rangle$$

$\langle \phi' | e^{-a\hat{H}} | \phi \rangle$ (Exercise)

$$= \left(\sqrt{\frac{1}{2\pi}} \right)^{N_1} \exp\left(- \sum_{n_1} a \left[\frac{1}{2} a \left(\frac{\phi'_{n_1} - \phi_{n_1}}{a} \right)^2 + \frac{1}{2} a \left(\frac{\phi_{n_1+1} - \phi_{n_1}}{a} \right)^2 + a V(\phi_{n_1}) \right] \right)$$

$$A_a = \int D\phi \, e^{-S_{E,a}(\phi)}$$

$$\int D\phi := \int \prod_{n_0=1}^{N_0-1} \prod_{n_1} d\phi_{n_0, n_1} \left(\sqrt{\frac{1}{2\pi}} \right)^{N_0 N_1}$$

$$S_{E,a} = \sum_{n_0} \sum_{n_1} a^2 \left(\frac{1}{2} \left(\frac{\phi_{n_0+1, n_1} - \phi_{n_0, n_1}}{a} \right)^2 + \frac{1}{2} \left(\frac{\phi_{n_0, n_1+1} - \phi_{n_0, n_1}}{a} \right)^2 + V(\phi_{n_0, n_1}) \right)$$

Naive continuum limit

$$\phi(\tau, x) \quad : \quad \phi(an_0, an_1) = \phi_{n_0, n_1}$$

$$S_E(\phi) = \int_0^\beta d\tau \int dx \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + V(\phi) \right)$$

※ n_0, n_1 are treated equally:

$n_0 \downarrow \tau$, $n_1 \downarrow x$

both of them are labels of the integration variables.

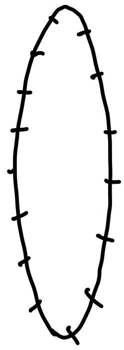
※ In Lorentzian path integral, t, x are equally treated.

$$S(\phi) = \int dt \int dx \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi) \right)$$

※ Here, we employ so-called "lattice regularization". Some other regularization, such as Pauli-Villars regularization, dimensional regularization, ... are also used in text books.

※ Regularization is <u>always</u> needed, though they are often implicit.

☐ **Eg.** 1d quantum Ising with transverse field.



$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{N_1}$$

$$\mathcal{H}_{n_1} = \mathbb{C}^2 \quad (\text{spin})$$

$$|+1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\sigma}^3 |\sigma\rangle = \sigma |\sigma\rangle \qquad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{H} = -\sum_{n_1} \left(J\, \hat{\sigma}^3_{n_1+1} \hat{\sigma}^3_{n_1} + K \hat{\sigma}^1_{n_1} \right)$$

$$\langle \sigma' | e^{-a\hat{H}} | \sigma \rangle$$

$$\approx \langle \sigma' | e^{+a\sum_{n_1} K \hat{\sigma}^1_{n_1}} \, e^{+a\sum_{n_1} J \hat{\sigma}^3_{n_1+1} \hat{\sigma}^3_{n_1}} | \sigma \rangle$$

Eigenvector of $\hat{\sigma}^3_{n_1}$

$$= \underbrace{\langle \sigma' | e^{+a\sum_{n_1} K \hat{\sigma}^1_{n_1}} | \sigma \rangle}_{\substack{N_1 \text{ copies of spins} \quad \hat{H} = -K\hat{\sigma}^1 \to 1d \text{ Ising}}} \, e^{a\sum_{n} J \sigma_{n_1+1} \sigma_{n_1}}$$

$$= \exp\left[\sum_{n_1} \left(J' \sigma'_{n_1} \sigma_{n_1} + J'' \sigma_{n_1+1} \sigma_{n_1} \right) \right]$$

(σ independent terms are ignored)

$$\langle \sigma_F | e^{-\beta \hat{H}} | \sigma_I \rangle$$

$$= \sum_{\{\sigma\}} \exp\Big[\sum_{n_0, n_1} \big(J' \sigma_{n_0+1, n_1} \sigma_{n_0, n_1} + J'' \sigma_{n_0, n_1+1} \sigma_{n_1} \big) \Big]$$

$\Rightarrow$ Classical 2d Ising model

$J' = J''$ $\Rightarrow$ space and Euclidean time are treated more equally.

※ Lorentzian path integral is similarly obtained.

3. Correlation Functions

operator formalism

path integral
classical statistical mechanics

$\langle 0 | \hat{A} | 0 \rangle$

an operator

$$\langle A \rangle = \frac{1}{Z} \int D\phi \, A \, e^{-S_E(\phi)}$$

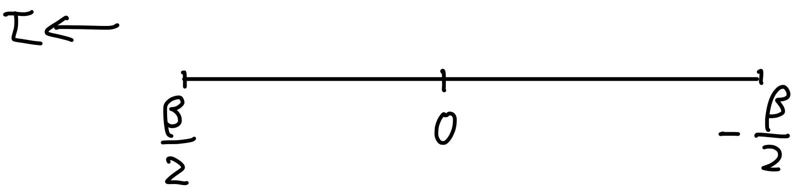
☆ ∞ interval

So far $\quad \langle \phi_F | e^{-\beta \hat{H}} | \phi_I \rangle = \int D\phi \, e^{-S_E(\phi)}$

$$S_E(\phi) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \; L_E$$

(τ is shifted)

Regularization is implicit.



□ $\beta \to \infty$

$$\langle \phi_F | e^{-\beta \hat{H}} | \phi_I \rangle$$

$$= \sum_n \langle \phi_F | n \rangle \langle n | e^{-\beta \hat{H}} | \phi_I \rangle$$

$$= \sum_n e^{-\beta E_n} \langle \phi_F | n \rangle \langle n | \phi_I \rangle$$

$$\to e^{-\beta E_0} \langle \phi_F | 0 \rangle \langle 0 | \phi_I \rangle \quad \cdots (*)$$

$|n\rangle$: Energy eigenstates

Assume ground state ("vacuum") is unique.

Assume $\langle \phi_F | 0 \rangle, \langle 0 | \phi_I \rangle \neq 0$

☐ Want to consider expectation value

$$\langle \phi_F | e^{-\frac{\beta}{2}\hat{H}} \hat{A} e^{-\frac{\beta}{2}\hat{H}} | \phi_I \rangle$$

$$= \sum_{n_F, n_I} \langle \phi_F | n_F \rangle \langle n_F | e^{-\frac{\beta}{2}\hat{H}} \hat{A} e^{-\frac{\beta}{2}\hat{H}} | n_I \rangle \langle n_I | \phi_I \rangle$$

$$= \sum_{n_F, n_I} e^{-\frac{\beta}{2}E_{n_F} - \frac{\beta}{2}E_{n_I}} \langle \phi_F | n_F \rangle \langle n_F | \hat{A} | n_I \rangle \langle n_I | \phi_I \rangle$$

$\beta \to \infty$ Only $n_I = n_F = 0$ survive

$$\longrightarrow e^{-\beta E_0} \langle \phi_F | 0 \rangle \langle 0 | \hat{A} | 0 \rangle \langle 0 | \phi_I \rangle$$

Combined with (*)

$$\Rightarrow \frac{\langle \phi_F | e^{-\frac{\beta}{2}\hat{H}} \hat{A} e^{-\frac{\beta}{2}\hat{H}} | \phi_I \rangle}{\langle \phi_F | e^{-\beta\hat{H}} | \phi_I \rangle} \longrightarrow \langle 0 | \hat{A} | 0 \rangle$$

Vacuum expectation value (VEV) of $\hat{A}$

※ Independent of ϕ_I, ϕ_F as far as

$$\langle \phi_F | 0 \rangle, \langle 0 | \phi_I \rangle \neq 0$$

$\square$ $\langle \phi_F | e^{-\frac{\beta}{2}\hat{H}} \hat{A} e^{-\frac{\beta}{2}H} | \phi_I \rangle$ in terms of PI

$$= \sum_{\phi_0, \phi_0'} \underbrace{\langle \phi_F | e^{-\frac{\beta}{2}\hat{H}} | \phi_0' \rangle}_{\text{done}} \langle \phi_0' | \hat{A} | \phi_0 \rangle \underbrace{\langle \phi_0 | e^{-\frac{\beta}{2}\hat{H}} | \phi_I \rangle}_{\text{done}}$$

$$= \sum_{\phi_0, \phi_0'} \sum_{\phi_1, \cdots, \phi_{\frac{N}{2}}} \sum_{\phi_{-1}, \cdots, \phi_{-\frac{N}{2}}} \times$$

$$\times e^{-S_{E,+}(\phi)} \langle \phi_0' | \hat{A} | \phi_0 \rangle e^{-S_{E,-}(\phi)} \quad \cdots (*)$$

$$S_{E,+}(\phi) = \sum_{n=0}^{\frac{N}{2}-1} a L_{\phi_{n+1}\phi_n} \quad , \quad \phi_0 = \phi_0'$$

$$S_{E,-}(\phi) = \sum_{n=-\frac{N}{2}}^{-1} a L_{\phi_{n+1}\phi_n} \quad , \quad \phi_0 = \phi_0$$

When

$\hat{A} = \hat{\phi}_i$ (a component, a label of site in the space)

$$\langle \phi_0' | \hat{\phi}_i | \phi_0 \rangle = \phi_{0,i} \, \delta_{\phi_0' \phi_0}$$

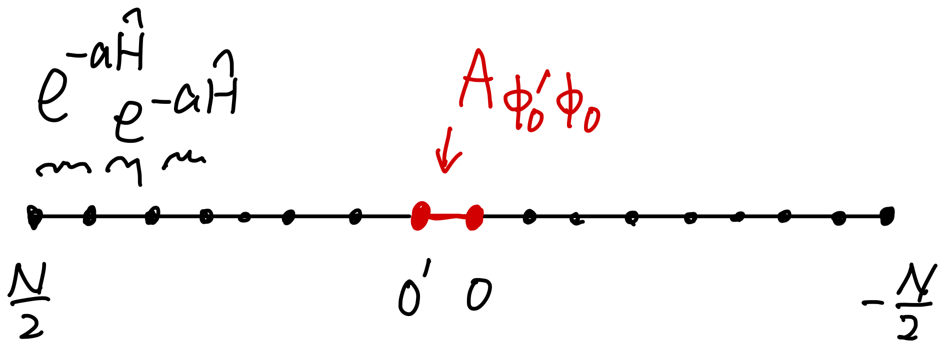
$$(*) = \sum_{\{\phi\}} \phi_{0,i} \, e^{-S_{E,a}(\phi)} = \int D\phi \, \phi_i(0) \, e^{-S_E(\phi)}$$

$\phi_{\frac{N}{2}} = \phi_F$ $\left(\phi_n, n = -\frac{N}{2}+1, \cdots, \frac{N}{2}-1 \right)$ (Regularization is implicit)

$\phi_{-\frac{N}{2}} = \phi_I$

For general $\hat{A}$

(*) is written as $\int D\phi \, A(0) \, e^{-S_E(\phi)}$



Sometimes called a "defect"

Take $\beta \to \infty$ limit

$\Rightarrow$ You can forget ϕ_I, ϕ_F boundary condition

$$\boxed{\begin{aligned} \langle 0 | \hat{A} | 0 \rangle &= \frac{\int D\phi \, A(0) \, e^{-S_E(\phi)}}{\int D\phi \, e^{-S_E(\phi)}} \\ \overset{..}{\vphantom{x}}\!\!\shortparallel \quad & \\ \langle A \rangle \quad & \end{aligned}}$$

☆ Correlation function

An Euclidean analogue of the Heisenberg operator

$$\hat{A}(\tau) := e^{\tau\hat{H}}\hat{A}e^{-\tau\hat{H}}$$

※ $\hat{A}(\tau)^\dagger = e^{-\tau\hat{H}}\hat{A}e^{\tau\hat{H}}$ $\qquad (\hat{A}^\dagger = \hat{A})$

$$= \hat{A}(-\tau)$$

> $\dagger$ is not intuitive in Euclidean formalism.
> Back to operator formalism if you want to consider $\dagger$

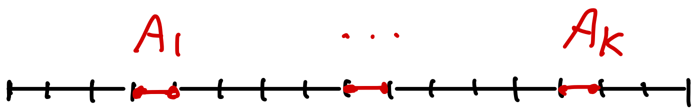
$\tau_1, \cdots, \tau_K$ s.t. $\frac{\beta}{2} > \tau_1 > \cdots > \tau_K > -\frac{\beta}{2}$

consider $\langle\phi_F| e^{-\frac{\beta}{2}\hat{H}} \hat{A}_1(\tau_1)\cdots\hat{A}_K(\tau_K) e^{-\frac{\beta}{2}\hat{H}} |\phi_I\rangle$

$$= \langle\phi_F| e^{-\underbrace{(\frac{\beta}{2}-\tau_1)}_{\vee 0}\hat{H}} \hat{A}_1 e^{-\underbrace{(\tau_1-\tau_2)}_{\vee 0}\hat{H}} \hat{A}_2 \cdots$$

$$\cdots \hat{A}_K e^{-\underbrace{(\tau_K+\frac{\beta}{2})}_{\vee 0}\hat{H}} |\phi_I\rangle$$

$= (*)$



$\leftarrow$ cl. stat. mech.

We already knew how to manipulate when $\tau = 0$.

Generic τ is similar.

$$(*) = \int_{B.C.} D\phi \, A_1(\tau_1) \cdots A_k(\tau_k) \, e^{-S_E(\phi)}$$

For generic order of τ_i

$\Rightarrow$ (Euclidean) time ordering

$$T(\hat{A}_1(\tau_1) \cdots \hat{A}_k(\tau_k)) := \hat{A}_{i_1}(\tau_{i_1}) \cdots \hat{A}_{i_k}(\tau_{i_k})$$

$$\text{s.t. } \tau_{i_1} > \tau_{i_2} > \cdots > \tau_{i_k}$$

choose $\beta > 0$ large enough so that

$$\frac{\beta}{2} > \tau_i > -\frac{\beta}{2}$$

$$\langle \phi_F | e^{-\frac{\beta}{2}\hat{H}} \, T(\hat{A}_1(\tau_1) \cdots \hat{A}_k(\tau_k)) \, e^{-\frac{\beta}{2}\hat{H}} | \phi_I \rangle$$

$$= \int_{B.C.} D\phi \, A_1(\tau_1) \cdots A_k(\tau_k) \, e^{-S_E(\phi)}$$

$$\boxed{\beta \to \infty \quad \langle 0 | T(\hat{A}_1(\tau_1) \cdots \hat{A}_k(\tau_k)) | 0 \rangle = \frac{\int D\phi \, A_1(\tau_1) \cdots A_k(\tau_k) \, e^{-S_E(\phi)}}{\int D\phi \, e^{-S_E(\phi)}}}$$

☆ Lorentzian formulation

$i\varepsilon$ prescription

① $\varepsilon > 0$, small

$T > 0$, large

consider
$$\langle \phi_F | e^{-iT(1-i\varepsilon)\hat{H}} | \phi_I \rangle$$
$$= \langle \phi_F | e^{-\varepsilon T\hat{H}} e^{-iT\hat{H}} | \phi_I \rangle$$

"Heisenberg operator" with $i\varepsilon$. $t \in \mathbb{R}$
$$\hat{A}_{i\varepsilon}(t) := e^{it(1-i\varepsilon)\hat{H}} \hat{A} e^{-it(1-i\varepsilon)\hat{H}}$$

$\Downarrow$

Same way as Euclidean case

② $T \to \infty$
$$\langle \phi_F | e^{-iT(1-i\varepsilon)\hat{H}} | \phi_I \rangle$$
$$\longrightarrow e^{-iT(1-i\varepsilon)E_0} \langle 0|0\rangle \langle \phi_F|0\rangle \langle 0|\phi_I\rangle$$

$\Downarrow$

$$\langle 0| T\left(\hat{A}_{1,i\varepsilon}(t_1)\,\hat{A}_{2,i\varepsilon}(t_2)\cdots\hat{A}_{K,i\varepsilon}(t_K)\right)|0\rangle$$

$$= \underbrace{\frac{\int D\phi_{i\varepsilon}\, A_{1,i\varepsilon}(t_1)\cdots A_{K,i\varepsilon}(t_K)\, e^{iS_{i\varepsilon}}}{\int D\phi_{i\varepsilon}\, e^{iS_{i\varepsilon}}}}_{(\#)}$$

③ $\varepsilon \to 0$

Heisenberg operators

$$\langle 0| T\big(\overbrace{\hat{A}_1(t_1)\cdots\hat{A}_K(t_K)}\big)|0\rangle = \lim_{\varepsilon\to 0}\ (\#)$$

※ In QFT text books, almost all Lorentzian path integrals are those with $i\varepsilon$ prescription. However, $i\varepsilon$ is often implicit, i.e. rhs is simply written as

$$\frac{\int D\phi\ A_1(t_1)\cdots A_K(t_K)\, e^{iS(\phi)}}{\int D\phi\ e^{iS(\phi)}}$$

4. From Path Integral to Operator Formalism

Operator formalism
Hilbert space
$\mathcal{H}$
Hamiltonian
$\hat{H}$

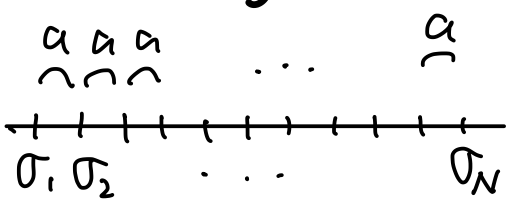
$\xrightarrow{\text{finished.}}$

$\xleftarrow[\text{now}]{}$

Path integral
cl. stat. mech.
$\int D\phi\, e^{-S_E(\phi)}$

Key word : "transfer matrix"

☆ 1 dim Ising model



1d lattice,
periodic boundary condition
(for simplicity)

$\sigma_n = \pm 1$

$$Z = \sum_{\{\sigma\}} \exp\left(K \sum_n \sigma_{n+1}\sigma_n \right)$$

$$= \sum_{\{\sigma\}} \exp(K\sigma_N\sigma_{N-1})\exp(K\sigma_{N-1}\sigma_{N-2})$$

$$\cdots \exp(K\sigma_2\sigma_1)\exp(K\sigma_1\sigma_K)$$

$$= \sum_{\{\sigma\}} T_{\sigma_N \sigma_{N-1}} \cdots T_{\sigma_2 \sigma_1} T_{\sigma_1 \sigma_N}$$

$$T_{\sigma'\sigma} := \exp(K\sigma'\sigma)$$

$\hat{T}$: operator on $\mathcal{H} = \mathbb{C}|\sigma=1\rangle \oplus \mathbb{C}|\sigma=-1\rangle$

$$\langle\sigma'|\hat{T}|\sigma\rangle = T_{\sigma'\sigma}$$

"transfer matrix"

$$= \mathrm{Tr}(\hat{T}^N)$$

Hilbert space $\mathcal{H}$

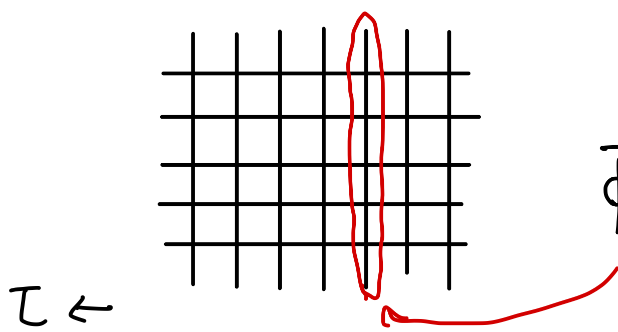
Hamiltonian $e^{-a\hat{H}} \approx \hat{T}$

$\Rightarrow$ Quantum mechanics (with cutoff)

☆ Generic argument

d-dim lattice, periodic boundary condition

① Choose Euclidean time direction



Φ : dof on a constant τ slice

② Find $T_{\bar{\phi}\bar{\phi}}$ s.t.

$$Z = \sum_{\{\phi\}} T_{\bar{\phi}_N \bar{\phi}_{N-1}} \cdots T_{\bar{\phi}_2 \bar{\phi}_1} T_{\bar{\phi}_1 \bar{\phi}_N}$$

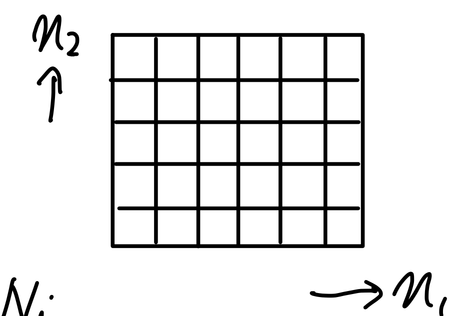
$$\Rightarrow Z = \mathrm{Tr}\, \hat{T}^N$$

$\mathcal{H}$: "linear combination" of $|\bar{\phi}\rangle$

$\hat{T}$: an operator on $\mathcal{H}$

s.t. $\hat{T}|\bar{\phi}\rangle = \sum_{\bar{\phi}'} |\bar{\phi}'\rangle T_{\bar{\phi}'\bar{\phi}}$

□ Eg. 2 dim Ising model



$$\sigma_{n_1, n_2} = \pm 1$$

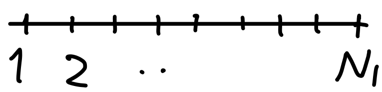
$n_i = 1, \cdots, N_i$

periodic boundary condition

$$Z = \sum_{\{\sigma\}} \exp\left(K \sum_{n_1, n_2} \left(\sigma_{n_1+1, n_2} \sigma_{n_1, n_2} + \sigma_{n_1, n_2+1} \sigma_{n_1, n_2} \right) \right)$$

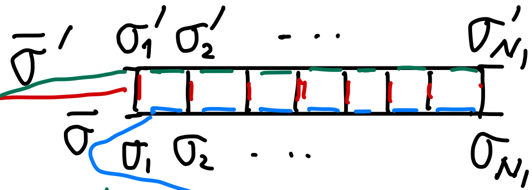
Let us choose n_2 as the Euclideantime direction

$$\Rightarrow \bar{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_{N_1})$$



$$T_{\bar\sigma'\bar\sigma} = \exp\Big(K \sum_{n_1} \sigma'_{n_1}\sigma_{n_1}$$

Vertical links



$$+ \frac{1}{2} K \sum_{n_1} \sigma'_{n_1+1}\sigma'_{n_1}$$

$\frac{1}{2}$ Horizontal links

$$+ \frac{1}{2} K \sum_{n_1} \sigma_{n_1+1}\sigma_{n_1} \Big)$$

$\frac{1}{2}$ Horizontal links

$\frac{1}{2}$ because both horizontal links of $\bar\sigma$ and $\bar\sigma'$.

$$\Rightarrow \quad Z = \mathrm{Tr}\left(\hat{T}^{N_2}\right)$$

5. Perturbation

☆ Problem

Path integral of regularized QFT

$$\int D\phi \, e^{-S_E(\phi)} \quad , \quad \phi^1, \phi^2, \cdots, \phi^N \in \mathbb{R}$$

$$\int D\phi := \int \prod_{i=1}^{N} \frac{d\phi^i}{\sqrt{2\pi}} \quad , \quad S_E : \mathbb{R}^N \longrightarrow \mathbb{C}$$

How to evaluate ?

- $S_E(\phi)$ is quadratic $\Rightarrow$ "exactly" evaluated as Gaussian integral

- $S_E(\phi)$ is close to quadratic

$\Rightarrow$ $\boxed{\text{Perturbation}}$

The most important technique in QFT

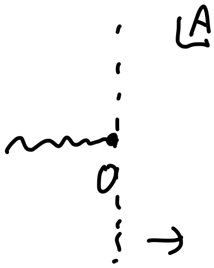
"Feynman diagram"

☆ Gaussian integral

$\phi \in \mathbb{R},\ A \in \mathbb{C},\ \mathrm{Re}\,A > 0$

① $$\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\, e^{-\frac{1}{2}A\phi^2} = \frac{1}{\sqrt{A}}$$

※ branch of $\sqrt{\ }$?



$\sqrt{\ }$ is holomorphic, singlevalued on $\mathrm{Re}\,A > 0$

s.t. $\sqrt{A} > 0,\ A > 0$

② $\phi^1, \cdots, \phi^N \in \mathbb{R}$ $\quad (A_i > 0)$

$$\underbrace{\int D\phi}_{\int \prod_i \frac{d\phi^i}{\sqrt{2\pi}}}\, e^{-\frac{1}{2}\sum_i A_i \phi^i \phi^i}$$

$$= \prod_i \left(\int_{-\infty}^{\infty} \frac{d\phi^i}{\sqrt{2\pi}}\, e^{-\frac{1}{2}A_i \phi^i \phi^i} \right) = \frac{1}{\sqrt{\prod_i A_i}}$$

③ A_{ij} : A_{ij} : real symmetric, positive definite (all the eigenvalues > 0)

$$S_0(\phi) = \frac{1}{2}A_{ij}\phi^i\phi^j$$

$$Z_0 = \int D\phi\, e^{-S_0(\phi)} = \frac{1}{\sqrt{\det A}}$$

$\because$

$$\exists U^i{}_j : \sum_k U^i{}_k U^j{}_k = \delta^{ij} \quad \Rightarrow \quad \det U = \pm 1$$

$$\text{s.t. } \sum_{i,j} A_{ij} U^i{}_k U^j{}_\ell = A_k \delta_{k\ell}$$

$$(A_k > 0) \ (\text{positive definite})$$

$$\tilde{\phi}^i : \quad \phi^i = \sum_k U^i{}_k \tilde{\phi}^k \quad \Rightarrow \quad \int D\phi = \int D\tilde{\phi}$$

$$\sum_{i,j} A_{ij} \phi^i \phi^j = \sum_{i,j,k,\ell} A_{ij} U^i{}_k U^j{}_\ell \tilde{\phi}^k \tilde{\phi}^\ell$$

$$= \sum_{k,\ell} A_k \delta_{k\ell} \tilde{\phi}^k \tilde{\phi}^\ell$$

$$= \sum_k A_k \tilde{\phi}^k \tilde{\phi}^k$$

$$Z_0 = \int D\tilde{\phi}\, e^{-\frac{1}{2}\sum_k A_k \tilde{\phi}^k \tilde{\phi}^k} = \frac{1}{\sqrt{\prod_k A_k}} = \frac{1}{\sqrt{\det A}} \qquad \blacksquare$$

---

④

A_{ij} : complex symmetric, $\underbrace{\text{Re}\, A_{ij} \text{ is positive definite}}_{(\#)}$

$$S_0(\phi) = \frac{1}{2} A_{ij} \phi^i \phi^j$$

$$Z_0 = \int D\phi\, e^{-S_0(\phi)} = \frac{1}{\sqrt{\det A}}$$

(phase is determined by analytic continuation)

☺ Use complex analysis

(i) $\int D\phi \, e^{-S_0(\phi)}$ are absolutely convergent

$$\left(\begin{array}{c} \left| e^{-S_0(\phi)} \right| = e^{-\frac{1}{2} \mathrm{Re} A_{ij} \phi^i \phi^j} \\ \text{is rapidly decreasing in } |\phi| \to \infty \end{array} \right)$$

(ii) Z_0 is holomorphic function of A_{ij} in the domain (#).

Thm: $D \subset \mathbb{C}$: domain, $f : \mathbb{R} \times D \to \mathbb{C}$, s.t.

(a) $f(t,z)$ is holomorphic wrt z

(b) $\forall K \subset D$, $\exists g(t)$, s.t $\underset{\text{for } \forall z \in K}{|f(t,z)| < g(t)}$, $\int_{-\infty}^{\infty} g(t)\,dt < +\infty$

"dominant function"

$\Rightarrow$ $F(z) = \int_{-\infty}^{\infty} f(t,z)\,dt$ is holomorphic in D

$$\frac{d}{dz} F(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z} f(t,z)\,dt$$

(iii) $Z_0 = \dfrac{1}{\sqrt{\det A}}$ if A_{ij} are real (3)

$\Rightarrow$ $Z_0 = \dfrac{1}{\sqrt{\det A}}$ for A_{ij} in (#)

(Identity theorem)

Expectation value

$$\langle \cdots \rangle_0 := \frac{1}{Z_0} \int D\phi \cdots e^{-S_0(\phi)}$$

$\uparrow$ function of ϕ^i's (polynomial of ϕ^i's)

How to evaluate ?

Generating function

$J_1, \cdots, J_N$: formal (real or complex) variables

$$Z_0(J) := \left\langle e^{\sum_i J_i \phi^i} \right\rangle_0$$

$$\frac{\partial}{\partial J_{i_1}} Z_0(J) = \left\langle \phi^{i_1} e^{\sum_i J_i \phi^i} \right\rangle_0$$

$$\frac{\partial}{\partial J_{i_1}} \frac{\partial}{\partial J_{i_2}} Z_0(J) = \left\langle \phi^{i_1} \phi^{i_2} e^{\sum_i J_i \phi^i} \right\rangle_0$$

$$\vdots$$

K-pt function

$$\langle \phi^{i_1} \phi^{i_2} \cdots \phi^{i_K} \rangle_0 = \frac{\partial}{\partial J_{i_1}} \frac{\partial}{\partial J_{i_2}} \cdots \frac{\partial}{\partial J_{i_K}} Z_0(J) \Big|_{J=0}$$

or

$$Z_0(J) = \sum_{K=0}^{\infty} \frac{1}{K!} \sum_{i_1, \cdots, i_K} \langle \phi^{i_1} \cdots \phi^{i_K} \rangle J_{i_1} \cdots J_{i_K}$$

Calculate $Z_0(J)$

$$Z_0(J) = \frac{1}{Z_0}\int D\phi\, e^{\underline{-S_0(\phi) + \sum_i J_i \phi^i}}$$

complete square

$$F(\phi) := -\frac{1}{2}\sum_{i,j} A_{ij}\phi^i\phi^j + \sum_i J_i\phi^i$$

Solve $\dfrac{\partial F(\phi)}{\partial \phi^i} = 0$

$$\Downarrow$$

$$-\sum_j A_{ij}\phi^j + J_i = 0 \quad \Rightarrow \sum_j A_{ij}\phi^j = J_i$$

solution $\quad \phi^i = \phi_{cl}^i := \Delta^{ij} J_j$

Δ^{ij} : inverse of A_{ij} $\quad A_{ij}\Delta^{jk} = \delta_i^k$

Change variable $\quad \phi^i \to \tilde{\phi}^i = \phi^i - \phi_{cl}^i$

$$F(\phi) = F(\phi_{cl} + \tilde{\phi})$$

$$= F(\phi_{cl}) + \underbrace{\frac{\partial F}{\partial \phi^i}(\phi_{cl})}_{=0}\tilde{\phi}^i + \frac{1}{2}\underbrace{\frac{\partial^2 F}{\partial \phi^i \partial \phi^j}(\phi_{cl})}_{-A_{ij}}\tilde{\phi}^i\tilde{\phi}^j$$

$$= F(\phi_{cl}) - S_0(\tilde{\phi})$$

$$Z_0(J) = \frac{1}{Z_0}\int D\tilde{\phi}\, e^{F(\phi_{cl}) - S_0(\tilde{\phi})} \qquad \left(\int D\phi = \int D\tilde{\phi}\right)$$

$$= e^{F(\phi_{cl})}\, \frac{1}{Z_0} \underbrace{\int D\tilde{\phi}\, e^{-S_0(\tilde{\phi})}}_{Z_0}$$

$$= e^{F(\phi_{cl})}$$

$$F(\phi_{cl}) = -\frac{1}{2} \underbrace{A_{ij}\, \phi_{cl}^{i}\, \phi_{cl}^{j}}_{J_i} + J_i\, \phi_{cl}^{i}$$

$$= -\frac{1}{2} J_i\, \phi_{cl}^{i} + J_i\, \phi_{cl}^{i}$$

$$= \frac{1}{2} \sum_i J_i\, \phi_{cl}^{i} \qquad \left(\phi_{cl}^{i} = \sum_j \Delta^{ij} J_j\right)$$

$$= \frac{1}{2} \Delta^{ij} J_i J_j$$

$$\boxed{Z_0(J) = e^{\frac{1}{2}\Delta^{ij} J_i J_j}}$$

Wick's theorem

$$\left(= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1, \cdots, i_k} \langle \phi^{i_1} \cdots \phi^{i_k} \rangle_0 J_{i_1} \cdots J_{i_k} \right)$$

$$Z_0(J) = \exp\left(\frac{1}{2} \sum_{i,j} \Delta^{ij} J_i J_j \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2} \sum_{i,j} \Delta^{ij} J_i J_j \right)^n$$

$$\left(\Rightarrow \text{If } k \text{: even}, \ \langle \phi^{i_1} \cdots \phi^{i_k} \rangle_0 = 0 \right)$$

$k = 2n$

$$\frac{1}{(2n)!} \sum_{i_1, \cdots, i_{2n}} \langle \phi^{i_1} \cdots \phi^{i_{2n}} \rangle_0 J^{i_1} \cdots J^{i_{2n}}$$

$$= \frac{1}{n!} \frac{1}{2^n} \sum_{i_1, \cdots, i_{2n}} \Delta^{i_1 i_2} \Delta^{i_3 i_4} \cdots \Delta^{i_{2n-1} i_{2n}}$$

$$\times J_{i_1} J_{i_2} \cdots J_{i_{2n-1}} J_{i_{2n}}$$

$\Downarrow$

$$\langle \phi^{i_1} \cdots \phi^{i_{2n}} \rangle = \frac{(2n)!}{2^n n!} \Delta^{(i_1 i_2} \cdots \Delta^{i_{2n-1} i_{2n})}$$

totally symmetrize

$$= \frac{(2n)!}{2^n n!} \frac{1}{(2n)!} \sum_{\sigma} \Delta^{i_{\sigma(1)} i_{\sigma(2)}} \cdots \Delta^{i_{\sigma(2n-1)} i_{\sigma(2n)}}$$

all permutations of $1, 2, \cdots, 2n$

Many identical terms

2^n exchange $\Delta^{ij} = \Delta^{ji}$

$n!$ permutations of Δ's

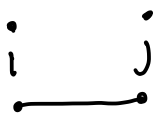
$$= \sum_{(a(1), b(1)) \cdots (a(n), b(n))} \Delta^{i_{a(1)} i_{b(1)}} \cdots \Delta^{i_{a(n)} i_{b(n)}}$$

all possible pairing of $1, \cdots, 2n$

$\boxed{\text{Wick's theorem}}$

Eg:

$$\langle \phi^i \phi^j \rangle_0 = \Delta^{ij} =:$$



$$\langle \phi^i \phi^j \phi^k \phi^l \rangle_0 = \Delta^{ij} \Delta^{kl} + \Delta^{ik} \Delta^{jl} + \Delta^{il} \Delta^{jk}$$

※ This is just a nice way to organize a Gaussian integral. No physical interpretation.

※ Label i may be a combination of several variables

<u>Eg.</u> $i = (\alpha, x)$

$\alpha = 1, 2, \quad x = (x^1, \cdots, x^d)$

$x^\mu = a, 2a, \cdots, N a$

ϕ^i is often written as $\phi^\alpha(x)$

The above argument is generic enough to include such cases.

※ Finding Δ^{ij} is a non-trivial task.

<u>Eg.</u> d-dim real scalar field

$\phi(x)$

$$S_{E,0}(\phi) = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 \right)$$

(regularization is implicit)

$$= \frac{1}{2} \int d^d x \int d^d y \; A(x, y) \, \phi(x) \, \phi(y)$$

$$A(x,y) = \partial_{x^\mu}\partial_{y^\mu}\delta^d(x-y) + m^2\delta^d(x-y)$$

$$\Rightarrow \quad \langle \phi(x)\phi(y)\rangle = \Delta(x,y)$$

$$\Delta : \quad \int d^dy\, A(x,y)\,\Delta(y,z) = \delta^d(x-z)$$

Find Δ

$$\int d^dy\, A(x,y)\,\Delta(y,z) = \int d^dy \Big(\overset{= -\partial_{y^\mu}}{\partial_{x^\mu}}\,\partial_{y^\mu}\delta^d(x-y) + m^2\delta^d(x-y)\Big)\Delta(y,z)$$

$$= \int d^dy\, \delta^d(x-y)\left(-\partial_y^2 + m^2\right)\Delta(y,z)$$

$$= \left(-\partial_x^2 + m^2\right)\Delta(x,z)$$

$$\left(-\partial_x^2 + m^2\right)\Delta(x,z) = \delta^d(x-z)$$

$$\Delta(x,0) =: \Delta(x)$$

$$\left(-\partial^2 + m^2\right)\Delta(x) = \delta^d(x) \quad \cdots\cdots (*)$$

$$x \to x - z \qquad \Rightarrow \quad \Delta(x,z) = \Delta(x-z)$$

Solve $(*)$

Fourier transform $\qquad \Delta(x) = \int_p e^{ipx}\,\hat{\Delta}(x)$

$$\int_p := \int \frac{d^dp}{(2\pi)^d}$$

$$\delta^d(x) = \int_p e^{ipx}$$

(*) $$\text{lhs} = (-\partial^2 + m^2)\int_p e^{ipx}\tilde{\Delta}(p)$$

$$= \int_p (-\partial^2 + m^2) e^{ipx}\tilde{\Delta}(p)$$

$$= \int_p e^{ipx}(p^2 + m^2)\tilde{\Delta}(p)$$

$$\text{rhs} = \int_p e^{ipx}$$

$$\Rightarrow \quad (p^2+m^2)\tilde{\Delta}(p) = 1 \quad \Rightarrow \quad \tilde{\Delta}(p) = \frac{1}{p^2+m^2}$$

$$\boxed{\Delta(x) = \int_p \frac{e^{ipx}}{p^2+m^2}}$$

- Another nice point of view

$$\phi(x) = \int_p e^{ipx}\tilde{\phi}(p)$$

$$S_0 = \int d^dx \left(\frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2\phi^2\right)$$

$$= \int d^dx \int_p \int_q \left(\frac{1}{2}\partial_\mu e^{ipx}\,\partial_\mu e^{iqx} + \frac{1}{2}m^2 e^{ipx} e^{iqx}\right)$$

$$\times\, \tilde{\phi}(p)\,\tilde{\phi}(q)$$

$$= \int d^dx \int_p \int_q \frac{1}{2}\left(-p\cdot q + m^2\right) e^{i(p+q)x}\, \tilde{\phi}(p)\tilde{\phi}(q)$$

$$\to \tilde{\delta}(p+q) := (2\pi)^d \delta^d(p+q)$$

$$= \int d^q x\, \frac{1}{2}\left(p^2+m^2\right) \tilde{\delta}(p+q)\, \tilde{\phi}(p)\tilde{\phi}(q)$$

$$\tilde{A}(p,q) := \left(p^2+m^2\right)\tilde{\delta}(p+q)$$

$\to$ inverse $\quad \tilde{\Delta}(p,q) = \dfrac{1}{p^2+m^2}\,\tilde{\delta}(p+q)$

$$\langle \tilde{\phi}(p)\tilde{\phi}(q)\rangle_0 = \tilde{\Delta}(p,q) = \frac{1}{p^2+m^2}\tilde{\delta}(p+q)$$

This "momentum basis" is often more convenient.

※ p, q are often called "momentum", but they are not physical momenta.

※ $\langle \phi(x)\phi(y)\rangle_0 = \int_p \int_q e^{ipx+iqy} \langle \tilde{\phi}(p)\tilde{\phi}(q)\rangle_0$

$$= \int_p \int_q e^{ipx+iqy} \frac{1}{p^2+m^2}\tilde{\delta}(p+q)$$

$$= \int_p e^{ip(x-y)} \frac{1}{p^2+m^2}$$

$$= \Delta(x-y)$$

☆ Introducing interaction

$$S(\phi) = S_0(\phi) + S_{int}(\phi)$$

$S_{int}(\phi) \propto \lambda$ small parameter (for counting order)

$$S_0(\phi) = \frac{1}{2} A_{ij} \phi^i \phi^j \quad ,$$

Eg. $$S_{int}(\phi) = \frac{1}{4!} V_{ijkl} \phi^i \phi^j \phi^k \phi^l$$

V_{ijkl} : totally symmetric $\propto \lambda$

$F(\phi)$: polynomial of ϕ

$$\langle F(\phi) \rangle := \frac{1}{Z} \int D\phi \, F(\phi) \, e^{-S(\phi)}$$

$$Z := \int D\phi \, e^{-S(\phi)}$$ "partition function"

Want to evaluate $\langle F(\phi) \rangle$ for small λ

"Perturbation" Power series of λ

□ Pertition function

$$Z = \int D\phi \, e^{-S(\phi)} = \int D\phi \, e^{-S_{int}(\phi)} e^{-S_0(\phi)}$$

$$= Z_0 \langle e^{-S_{int}(\phi)} \rangle_0$$

($\langle \cdots \rangle_0 := \frac{1}{Z_0} \int D\phi \cdots e^{-S_0(\phi)}$ Gaussian integral (free theory))

$$\frac{Z}{Z_0} = \left\langle e^{-S_{int}(\phi)} \right\rangle_0 = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \left(-S_{int}(\phi)\right)^n \right\rangle_0$$

$$\left(\cancel{=} \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(-S_{int}(\phi)\right)^n \right\rangle_0 \right)$$

$$= \left\langle \sum_{n=0}^{N} \frac{1}{n!} \left(-S_{int}(\phi)\right)^n + O(\lambda^{N+1}) \right\rangle_0$$

$$= \sum_{n=0}^{N} \frac{1}{n!} \left\langle \left(-S_{int}(\phi)\right)^n \right\rangle_0 + O(\lambda^{N+1})$$

↑ Evaluate by Wick's theorem

Key to understanding:

Don't consider all the terms at the same time.

Look at each term one by one.

Eg. $S_{int} = \frac{1}{4!} V_{ijkl} \phi^i \phi^j \phi^k \phi^l$

$$\frac{1}{n!} \left\langle \left(-S_{int}(\phi)\right)^n \right\rangle_0$$

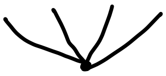
$$= \frac{1}{n!} \left(\frac{1}{4!}\right)^n \Big\langle \overbrace{(-V_{i_1 j_1 k_1 l_1}) \phi^{i_1} \phi^{j_1} \phi^{k_1} \phi^{l_1}}$$

$$\times (-V_{i_2 j_2 k_2 l_2}) \phi^{i_2} \phi^{j_2} \phi^{k_2} \phi^{l_2} \cdots \Big\rangle_0$$

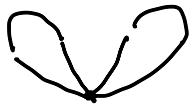
Eg. $n = 0$

$$\langle 1 \rangle_0 = 1$$

Eg. $n=1$



contraction

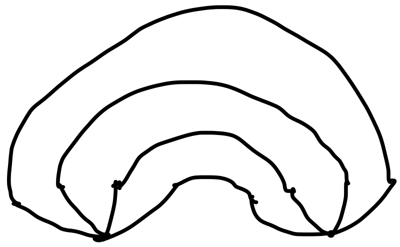


$\sim (-V_{ijkl}) \Delta^{ij} \Delta^{kl}$

$+ \cdots$

$n=2$

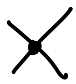
contraction



$+ \cdots$

$\sim (-V_{i_1 j_1 k_1 l_1})(-V_{i_2 j_2 k_2 l_2})$
$\times \Delta^{i_1 i_2} \Delta^{j_1 j_2} \Delta^{k_1 k_2} \Delta^{l_1 l_2}$

$\Rightarrow$ Diagrams made by

 and —

"Feynman diagram"

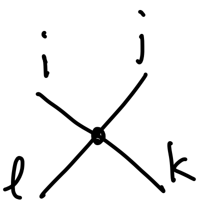
※ $n=0$ $\quad 1$ = "empty diagram"

$$\frac{Z}{Z_0} = \sum_{\substack{\text{all diagrams } I \\ \text{up to } \lambda^N}} \frac{1}{S_I} \hat{D}_I + O(\lambda^{N+1})$$

combinatrial factor independent of Δ, V "symmetry factor"

contribution of V, Δ

First consider $\hat{D}_I$

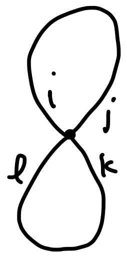


$= -V_{ijkl}$,

$= \Delta^{ij}$

"Feynman rule"

Eg.



$$\Rightarrow -V_{ijkl}\,\Delta^{ij}\,\Delta^{kl}$$

We want to discuss symmetry factor later

Correlation function

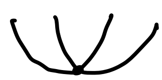
$$\langle F(\phi)\rangle = \frac{1}{Z}\int D\phi\, F(\phi)\, e^{-S(\phi)}$$

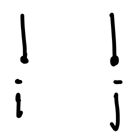
$$= \frac{Z_0}{Z}\left\langle F(\phi)\, e^{-S_{int}(\phi)}\right\rangle_0$$

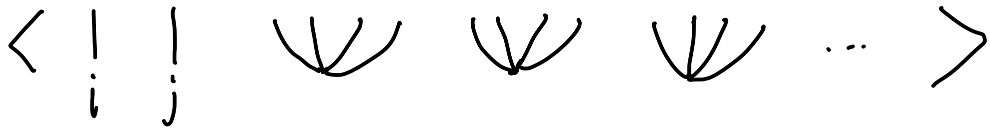
$$= \frac{\left\langle F(\phi)\, e^{-S_{int}(\phi)}\right\rangle_0}{\left\langle e^{-S_{int}(\phi)}\right\rangle_0}$$

$$\left\langle F(\phi)\, e^{-S_{int}(\phi)}\right\rangle_0$$

$$= \cdots + \frac{1}{n!}\left\langle F(\phi)\left(-S_{int}(\phi)\right)^n\right\rangle_0 + \cdots$$

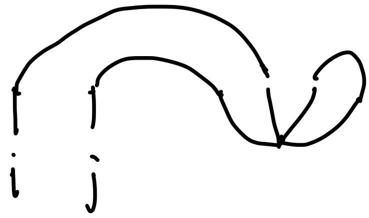
Eg. $S_{int}(\phi) = \frac{1}{4!} V_{ijkl}\,\phi^i\phi^j\phi^k\phi^l \rightarrow$ 

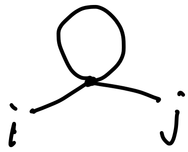
$F(\phi) = \phi^i\phi^j \rightarrow$ 

$= \cdots \frac{1}{n!}$ 

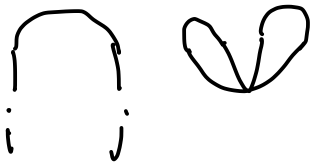
<u>Eg.</u> $n=1$

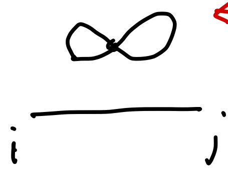
contraction

$\Rightarrow$ 

$\Rightarrow$ 

another contraction

$\Rightarrow$ 

$\Rightarrow$ 

← vacuum bubble

$\Rightarrow$ Feynman diagrams with external lines

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_k}\, e^{-S_{int}(\phi)} \rangle_0$$

$$= \sum_{\substack{\text{diagrams}\\ \text{external lines}\\ i_1,\cdots,i_k}} \frac{1}{S_I}\hat{D}_I + O(\lambda^{N+1})$$

up to $O(\lambda^N)$

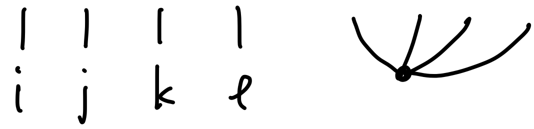
Symmetry factor

$$S_{int}(\phi) = \frac{1}{4!} V_{ijkl}\, \phi^i \phi^j \phi^k \phi^l$$

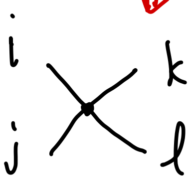
determined so that a "generic diagram" has the symmetry factor 1

Eg. $\langle \phi^i \phi^j \phi^k \phi^l \, e^{-S_{int}(\phi)} \rangle_0$

$= \cdots + \frac{1}{4!}$



$\Downarrow$



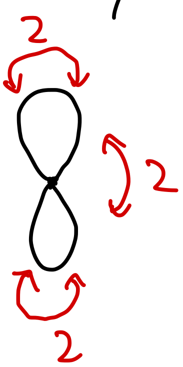
4! ways of contraction. to get this diagram.

$\Rightarrow \frac{1}{S} = 1$

$\Rightarrow$ In general, S = (order of the symmetry group of the diagram)

symmetry factor $= \frac{1}{S}$

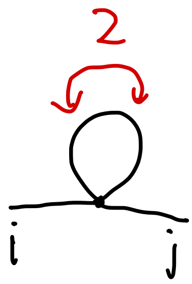
Eg.



$\Rightarrow \frac{1}{8}(-V \dots)\, \Delta^{\cdot\cdot}\, \Delta^{\cdot\cdot}$

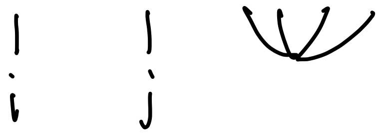
check. $\frac{1}{4!}$  $\Rightarrow$ 3 ways of cont. to get 

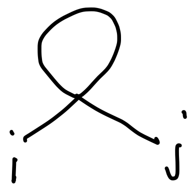
$$\frac{1}{S} = \frac{3}{4!} = \frac{1}{8} \quad \text{O.K.}$$

Eg. 

$\Rightarrow \frac{1}{2}(-V\ldots)\Delta^{\cdot i}\Delta^{\cdot j}$

check

$\frac{1}{4!}$ 

$\Rightarrow$ $4\cdot 3$ ways to get 

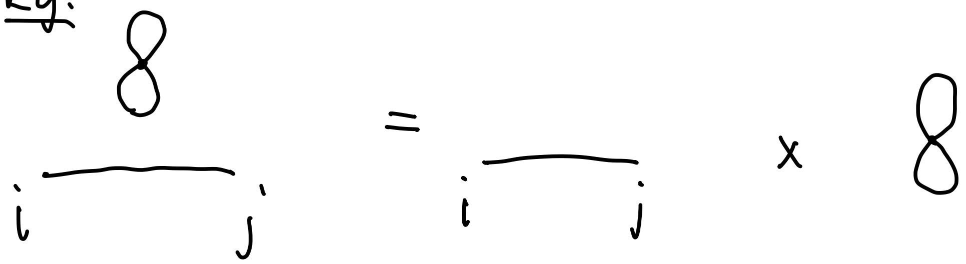
$$\frac{4\cdot 3}{4!} = \frac{1}{2}$$

※ You can easily make a mistake when finding the symmetry group.

In general, it is more reliable to count the ways of contraction.

Vacuum bubble

Eg.



(including symmetry factor)

$$\langle \phi^{i_1} \cdots \phi^{i_k} e^{-S_{int}(\phi)} \rangle_0$$

include $\frac{1}{S_I}$

$$= \sum_{\text{all diag.}} D_I + O(\lambda^{N+1})$$

$$= \Big(\sum_{\substack{\text{all diag} \\ \text{without} \\ \text{vacuum}}} D_I \Big) \underbrace{\Big(\sum_{\substack{\text{all} \\ \text{vac.} \\ \text{bub.}}} D_I \Big)}_{\langle e^{-S_{int}(\phi)} \rangle_0 + O(\lambda^{N+1})} + O(\lambda^{N+1})$$

$\Downarrow$

$$\langle \phi^{i_1} \cdots \phi^{i_k} \rangle = \frac{\langle \phi^{i_1} \cdots \phi^{i_k} e^{-S_{int}(\phi)} \rangle_0}{\langle e^{-S_{int}(\phi)} \rangle_0}$$

$$= \sum D_I + O(\lambda^{N+1})$$

all diagrams with external lines $i_1, \cdots, i_k$ without vacuum bubble.

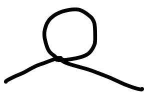
Eg:

$\phi \in \mathbb{R}$ single variable

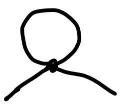
$$S(\phi) = \frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad \Rightarrow \quad Z = \int D\phi \, e^{-S(\phi)}$$

$\Rightarrow$

—	$= 1$
$\times$	$= -\lambda$

$\langle \phi \phi \rangle =$ — $+$  $+ \; O(\lambda^2)$

— $= 1$

 $= \frac{1}{2}(-\lambda) \quad \Leftarrow \quad \langle \phi \phi \, e^{-S_{int}(\phi)} \rangle_0$

$\uparrow$ symmetry factor

$$= \dots \; -\frac{\lambda}{4!}\langle \phi \phi \; \phi^4 \rangle$$

4x3 ways of contraction

$$\langle \phi \phi \rangle = 1 - \frac{1}{2}\lambda + O(\lambda^2)$$

Eg. d-dim ϕ^4 theory

$$S_E(\phi) = \int d^dx \left(\frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$

※ regularization is implicit. (Eg. dimensional regularization)

※ If you want to consider continuum limit, there is a more useful convention (renormalized perturbation)

$$\phi(x) = \int_p e^{ip\cdot x} \tilde{\phi}(p) \qquad \int_p := \int \frac{d^dp}{(2\pi)^d}$$

$$\tilde{\delta}(p) := (2\pi)^d \delta^d(p)$$

$$\Rightarrow \int_p \tilde{\delta}(p) = 1$$

$$S_E = S_{E,0} + S_{E,\mathrm{int}}$$

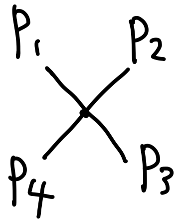
$$S_{E,0} = \frac{1}{2} \int_{p,q} (p^2+m^2)\, \tilde{\delta}(p+q)\, \tilde{\phi}(p)\, \tilde{\phi}(q) \qquad \text{(already done)}$$

$$= \frac{1}{p^2+m^2} \tilde{\delta}(p+q)$$

$$S_{E,\mathrm{int}} = \int d^dx \frac{\lambda}{4!} \phi(x)^4$$

$$= \frac{\lambda}{4!} \int d^dx \int_{p_1,\cdots,p_4} \underbrace{e^{i(p_1+p_2+p_3+p_4)x}}_{\tilde{\delta}(p_1+\cdots+p_4)} \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)$$

$$= \frac{1}{4!} \int_{p_1,\cdots,p_4} \lambda \, \tilde{\delta}(p_1+\cdots+p_4)\, \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_4)$$



$$= -\lambda \,\tilde{\delta}(p_1 + \cdots + p_4)$$

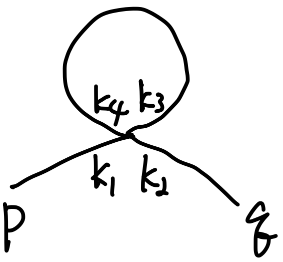
Some simplification

Lots of $\tilde{\delta}(\cdots) \leftarrow$ (translation invariance)

$\downarrow$

Eliminate integrations

<u>Eg.</u> a connected diagram symmetry factor



$$= \frac{1}{2} \int_{k_1} \int_{k_2} \int_{k_3} \int_{k_4} \times (-\lambda)\, \tilde{\delta}(k_1 + k_2 + k_3 + k_4)$$

$$\times \frac{1}{p^2 + m^2} \tilde{\delta}(p + k_1) \times \frac{1}{q^2 + m^2} \tilde{\delta}(q + k_2)$$

$$\times \frac{1}{k_4^2 + m^2} \tilde{\delta}(k_4 + k_3)$$

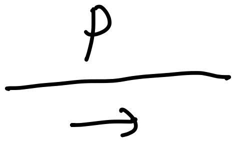
$$k_1 = -p, \; k_2 = -q, \; k_3 = -k_4, \; (k_4 =: k)$$

$$= \underbrace{\frac{1}{2}\int_k}_{③} \underbrace{(-\lambda)}_{②} \underbrace{\frac{1}{p^2+m^2}}_{①} \underbrace{\frac{1}{q^2+m^2}}_{①} \underbrace{\frac{1}{k^2+m^2}}_{①} \quad \underbrace{\tilde{\delta}(p+q)}_{④}$$

(k : loop momentum)

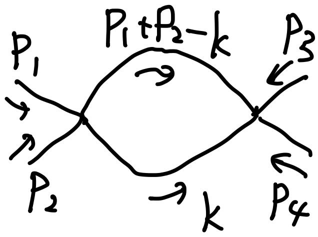
①

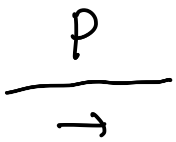
$\Longrightarrow$



assign momentum for each line with arrow

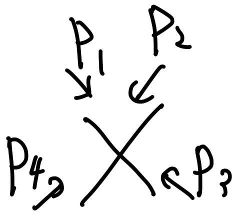
so that momenta are conserved at each vertex





$= \frac{1}{p^2+m^2}$

②



$= -\lambda$ (independent of momenta)

④ $\tilde{\delta}$(total incomming momenta)

Lorentzian correlation functions of ϕ^4 theory

$$S_L(\phi) = \int d^dx \left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right)$$

(Be careful about the sign)

$i\varepsilon$ prescription $\rightsquigarrow$ In perturbation, m^2 is replaced by $m^2 - i\varepsilon$

Path integral is $\int D\phi\, e^{iS_L(\phi)}$

$\mathrm{Re}\, A(x,y) = \varepsilon\, \delta^d(x-y)$ is positive definite

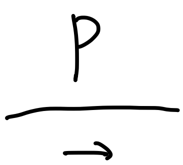
$$iS_L = iS_{L,0} + iS_{L,int}$$

$$iS_{L,0} = -\frac{1}{2}\int_{p,q} i(p^2+m^2-i\varepsilon)\,\tilde{\delta}(p+q)\,\tilde{\phi}(p)\,\tilde{\phi}(q)$$

$$(p^2 = \eta^{\mu\nu}p_\mu p_\nu)$$

$$\tilde{A}(p,q) = i(p^2+m^2-i\varepsilon)\,\tilde{\delta}(p+q)$$

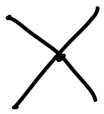
$$\Rightarrow \langle \tilde{\phi}(p)\,\tilde{\phi}(q)\rangle_0 = \frac{-i}{p^2+m^2-i\varepsilon}\,\tilde{\delta}(p+q)$$



$$= \frac{-i}{p^2+m^2-i\varepsilon}$$

$$iS_{L,\mathrm{int}} = \frac{1}{4!}\int_{p_1,\cdots,p_4} (-i\lambda)\,\tilde{\delta}(p_1+\cdots+p_4)\,\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_4)$$

$$\Downarrow$$



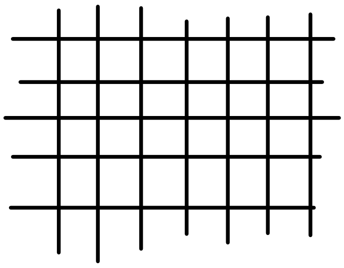
$$= -i\lambda$$

6. Renormalization Group

☆ Introduction

So far, cutoff theory (regularized theory)

Path integral $\sim$ finite dimensional integral, mathematically well-defined



How to get continuous QFT from this ?

Key concept : "Renormalization group" (K. G. Wilson)

- Not focus on a theory, but a class of theories the space of theories

- Universality : physics in the long range (relevant in continuum limit) compared to lattice spacing.

  $\Uparrow$

  fixed point theory, relevant deformation

  (scale invariant theory = conformal field theory)
  most of them

☆ Correlation length

cutoff theory on a lattice (eg. Ising model)

coordinates $n = (n_1, n_2, \cdots, n_d)$

$n_\mu \in \mathbb{Z}$

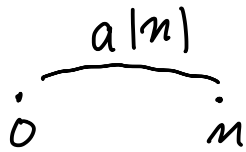
σ_n : dof at each site (such as spin)

two point function

$$\langle \sigma_n \sigma_0 \rangle \sim \exp\left(-\frac{|n|}{\xi}\right) \Rightarrow \xi : \text{"correlation length"}$$

dof within this region are correlated and cannot be separated.

dimensionful lattice spacing a



Continuum limit $a \to 0$

$a|n|$: dimensionful physical distance from 0 to n

phys. dist. finite

$$\langle \sigma_n \sigma_0 \rangle \sim \exp\left(-\frac{|n|}{\xi}\right) = \exp\left(-\frac{a|n|}{a\xi}\right)$$

If $a|n|$ fixed finite, $\langle \sigma_n \sigma_0 \rangle \to 0$,

unless $\xi \to \infty$

$\Rightarrow$ If ξ is finite in $a \to 0$, "empty" theory

no propagating dof. (topological field theory)

This is not our target.

To find QFT with propagating dof,

we have to find cutoff theory with $\xi = \infty$

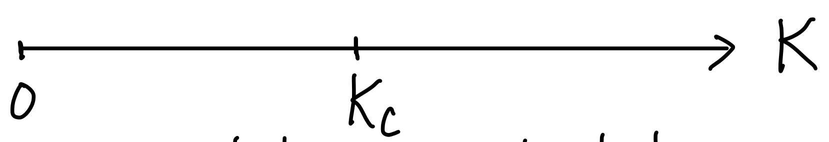
(or series of cutoff theories $\xi \to \infty$)

<u>Eg.</u> Ising model $d > 1$

$\sigma_n = \pm 1$

$$S(\sigma) = -K \sum_{n,\mu} \sigma_{n+\hat{\mu}} \sigma_n$$

$$\hat{\mu} = (0, \cdots, 0, \overset{\mu}{1}, 0, \cdots 0)$$



disordered phase
symmetric phase
"$\langle \sigma_n \rangle = 0$"

$$\lim_{|n| \to \infty} \langle \sigma_n \sigma_0 \rangle = 0$$

ordered phase
sponteneous symmetry breaking
"$\langle \sigma_n \rangle \neq 0$"

$$\lim_{|n| \to \infty} \langle \sigma_n \sigma_0 \rangle \neq 0$$

2nd order phase transition

In $K \to K_c - 0$, $\xi \to \infty$

Questions :

- What is K_c ? (not universal)

- How rapidly $\xi \to \infty$ in $K \to K_c$?

$$\xi \sim \frac{1}{(K_c - K)^\nu}$$ (ν: a critical exponent)

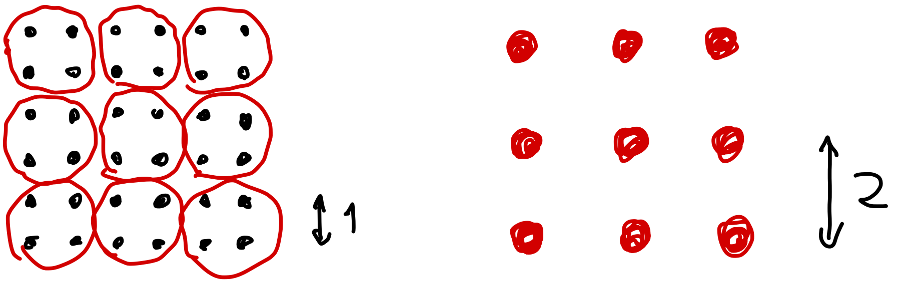
☆ First look at RG

RG : ① coarse graining , ② scaling.

Microscopic theory S_0 $\xi_0 \gg 1$

① Coarse graining (block spin transformation)

without changing the long range physics



2^d spins $\Rightarrow$ 1 spin

② Scaling . redefine unit length

new lattice spacing $= 1$

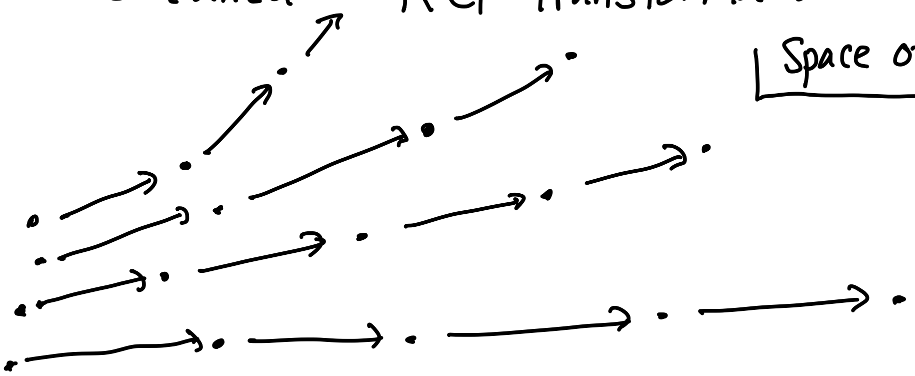
$\Rightarrow S_1$, $\xi_1 = \frac{1}{2}\xi_0$

Repeat $\Rightarrow$ S_n , $\xi_n = \frac{1}{2^n}\xi_0$

$$S_n \xrightarrow[①,②]{} S_{n+1}$$

is called "RG transformation"

Space of cutoff theories



"RG flow"

Fixed point

S^* : invariant by the RG transf.

$$S^* \longrightarrow S^*$$

Correlation length

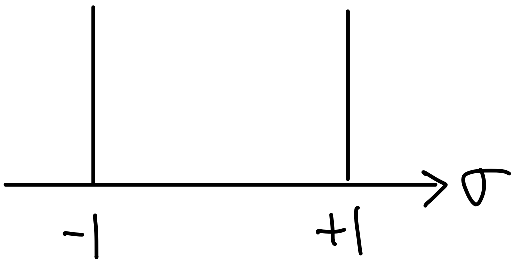
$\xi^* = \frac{1}{2}\xi^*$ $\Rightarrow$ $\xi^* = 0$ or ∞

"empty" "non-empty" QFT

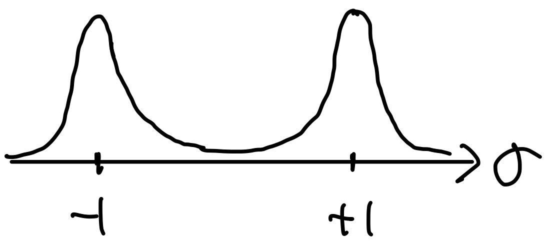
☆ Ising model and ϕ^4 theory

Rewriting Ising model

$$\sum_{\sigma=\pm} = \int_{-\infty}^{\infty} d\sigma \left(\underbrace{\delta(\sigma-1) + \delta(\sigma+1)} \right)$$



↑ limit $A \to \infty$ with a good choice of $\mathcal{N}(A)$



$$\mathcal{N} \exp\left(-A(\sigma^2-1)^2\right)$$

↑ specialize

$$\exp\left(-\tfrac{1}{2} b' \sigma^2 - u' \sigma^4 + c\right)$$

Ignore these constant terms for simplicity.

$$Z = \sum_{\{\sigma\}} \exp\left(K \sum_n \sum_\mu \sigma_{n+\hat{\mu}} \sigma_n\right)$$

$$= \int D\sigma \exp\left(-S(\sigma)\right)$$

$\prod_n d\sigma_n$

$$S(\sigma) = \sum_n \left[\frac{1}{2} b' \sigma_n^2 + u' \sigma_n^4 - K \sum_\mu \underbrace{\sigma_{n+\hat{\mu}} \sigma_n}_{-\frac{1}{2}(\sigma_{n+\hat{\mu}} - \sigma_n)^2 + \frac{1}{2}\sigma_n^2 + \frac{1}{2}\sigma_{n+\hat{\mu}}^2} \right]$$

$$= \sum_n \left[\frac{1}{2} K \sum_\mu (\sigma_{n+\hat{\mu}} - \sigma_n)^2 + \frac{1}{2} b \sigma_n^2 + u' \sigma_n^4 \right]$$

$$(b = b' - 2dK)$$

$$\phi_n := \sqrt{K}\, \sigma_n$$

$$S(\phi) = \sum_n \left[\frac{1}{2} \sum_\mu (\phi_{n+\hat{\mu}} - \phi_n)^2 + \frac{1}{2} r \phi_n^2 + \frac{u}{4!} \phi_n^4 \right]$$

$$\left(r = \frac{b}{K}, \quad \frac{u}{4!} = \frac{u'}{K^2} \right)$$

$$D\sigma = D\phi \underbrace{(\sqrt{K})^{-N}}_{\phi \text{ independent constant} \to \text{ignore}} \quad (\#\,\text{sites})$$

$$Z = \int D\phi \, e^{-S(\phi)}$$

Fourier transformation

$$\phi_n = \int_p e^{ip\cdot n} \tilde{\phi}_p \qquad (\text{Large volume} \to p \sim \text{continuous})$$

$$e^{i(p+2\pi\hat{\mu})\cdot n} = e^{ip\cdot n} e^{2\pi i n_\mu} = e^{ip\cdot n}$$

$$\Rightarrow \quad p \sim p + 2\pi\hat{\mu}$$

We choose 1st Brillouin zone $\quad -\pi \le p_\mu \le \pi$

(finite region)

$$\int_p := \prod_\mu \int_{-\pi}^{\pi} \frac{dp_\mu}{2\pi}$$

$$\Rightarrow \quad \int_p e^{ip\cdot n} = \delta_n$$

$$\sum_n e^{ip\cdot n} = \tilde{\delta}(p) := \sum_{k\in\mathbb{Z}^d} (2\pi)^d \delta^d(p+2\pi k)$$

$$\phi_{n+\hat{\mu}} - \phi_n = \int_p \left(e^{ip\cdot(n+\hat{\mu})} - e^{ip\cdot n}\right)\tilde{\phi}_p$$

$$= \int_p e^{ip\cdot n}\left(e^{ip_\mu} - 1\right)\tilde{\phi}_p$$

$$S_{kin} := \sum_n \frac{1}{2}\sum_\mu \left(\phi_{n+\hat{\mu}} - \phi_n\right)^2$$

$$= \underbrace{\sum_n \int_{p,q} e^{i(p+q)\cdot n}}_{\tilde{\delta}(p+q)} \frac{1}{2}\sum_\mu \left(e^{ip_\mu}-1\right)\left(e^{iq_\mu}-1\right)$$

$$\times \tilde{\phi}_p \tilde{\phi}_q$$

$$= \int_p \frac{1}{2} \sum_\mu \underbrace{(e^{ip_\mu} - 1)(e^{-ip_\mu} - 1)}_{2 - e^{ip_\mu} - e^{-ip_\mu}} \tilde{\phi}_p \tilde{\phi}_{-p}$$

$$= 2 - 2\cos p_\mu$$

$$= 4 \sin^2 \frac{p_\mu}{2}$$

$$= \int_p \frac{1}{2} \sum_\mu 4 \sin^2 \frac{p_\mu}{2} \tilde{\phi}_p \tilde{\phi}_{-p}$$

$$S = \int_p \frac{1}{2} \left(\sum_\mu 4 \sin^2 \frac{p_\mu}{2} + r \right) \tilde{\phi}_p \tilde{\phi}_{-p} \quad + S_{int}$$

$$S_{int} = \frac{u}{4!} \int_{p_1, \cdots, p_4} \tilde{\delta}(p_1 + p_2 + p_3 + p_4) \tilde{\phi}_{p_1} \tilde{\phi}_{p_2} \tilde{\phi}_{p_3} \tilde{\phi}_{p_4}$$

□ Some simplification

We simplify the theory without changing long range behavior.

I. $\sum_\mu 4 \sin^2 \frac{p_\mu}{2} \approx \sum_\mu p_\mu^2 = p^2$ for small p

$\sim$ large wave length

II. $\int_p = \prod_\mu \int_{-\pi}^{\pi} \frac{dp_\mu}{2\pi} \approx \int_{|p|<1} \frac{d^d p}{(2\pi)^d}$

$$\square \Rightarrow \bigcirc$$

III. $\tilde{\delta}(p) \approx (2\pi)^d \delta^d(p)$

☆ Gaussian model

Consider $S_{int} = 0$ case

$$S = \int_p \frac{1}{2}(p^2+r)\tilde{\phi}_p \tilde{\phi}_{-p} \quad , \quad \int_p := \int_{|p|<1} \frac{d^dp}{(2\pi)^d}$$

$$Z = \int D\tilde{\phi} \, \exp(-S(\tilde{\phi}))$$

$$D\tilde{\phi} = \prod_{|p|<1} d\tilde{\phi}_p \quad \left(\text{※} \quad \begin{array}{l} \tilde{\phi}_p^* = \tilde{\phi}_{-p} \\ d\tilde{\phi}_p \, d\tilde{\phi}_{-p} := \int d\,\mathrm{Re}\tilde{\phi}_p \, d\,\mathrm{Im}\tilde{\phi}_p \end{array} \right)$$

① Coase graining

= Integrate out $\tilde{\phi}_p$, $\frac{1}{2} \leq |p| < 1$

$$D\tilde{\phi}_< = \prod_{|p|<\frac{1}{2}} d\tilde{\phi}_p \, , \qquad D\tilde{\phi}_> = \prod_{|p|\geq\frac{1}{2}} d\tilde{\phi}_p$$

$$\Rightarrow D\tilde{\phi} = D\tilde{\phi}_> D\tilde{\phi}_<$$

$$S_< = \int_{|p|<\frac{1}{2}} \frac{1}{2}(p^2+r)\tilde{\phi}_p \tilde{\phi}_{-p}$$

$$S_> = \int_{|p|>\frac{1}{2}} \frac{1}{2}(p^2+r)\tilde{\phi}_p \tilde{\phi}_{-p}$$

$$S = S_> + S_< \qquad \left(\tilde{\phi}_p \, , \, \left. \begin{array}{l} |p| \geq \frac{1}{2} \\ |p| < \frac{1}{2} \end{array} \right\} \text{decouple} \right)$$

$$Z = \int D\tilde{\phi}_< \int D\tilde{\phi}_> \exp\left(-S_<(\tilde{\phi}) - S_>(\tilde{\phi})\right)$$

$$= \text{const} \int D\tilde{\phi}_< \exp\left(-S_<(\tilde{\phi})\right)$$

$$\left(\text{const} = \int D\tilde{\phi}_> \exp\left(-S_>(\tilde{\phi})\right) \rightsquigarrow \text{Ignore} \right)$$

$$= \int D\tilde{\phi}_< \exp\left(-S_<(\tilde{\phi})\right)$$

② Scaling

Dof $\tilde{\phi}_p \quad |p| < \frac{1}{2}$

$$p' := 2p \quad \Rightarrow \quad |p'| < 1$$

$$S_< = \int_{|p|<\frac{1}{2}} \frac{1}{2}(p^2 + r)\,\tilde{\phi}_p\,\tilde{\phi}_{-p}$$

$$= \int_{|p'|<1} 2^{-d}\,\frac{1}{2}\left(\left(\tfrac{1}{2}p'\right)^2 + r\right)\tilde{\phi}_{\frac{1}{2}p'}\,\tilde{\phi}_{-\frac{1}{2}p'}$$

$$= \int_p 2^{-d-2}\,\frac{1}{2}(p^2 + 2^2 r)\,\tilde{\phi}_{\frac{1}{2}p}\,\tilde{\phi}_{-\frac{1}{2}p}$$

$$\left(\tilde{\phi}'_p := 2^{-\frac{d+2}{2}}\,\tilde{\phi}_{\frac{1}{2}p}, \quad r' = 2^2 r \right)$$

$$= \int_p \frac{1}{2}(p^2 + r')\,\tilde{\phi}'_p\,\tilde{\phi}'_{-p} = S'(\tilde{\phi}')$$

$$D\tilde{\phi}_< = D\tilde{\phi}' \times \text{const}$$

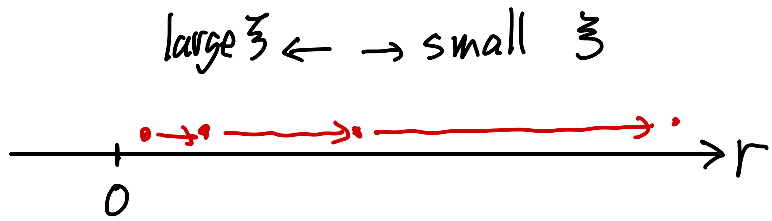
Ignore

$$Z = \int D\tilde{\phi}' \, e^{-S'(\tilde{\phi}')}$$

$$= \int D\tilde{\phi} \, e^{-S'(\tilde{\phi})}, \quad S'(\tilde{\phi}) = \int_p \frac{1}{2}(p^2 + r')\tilde{\phi}_p \tilde{\phi}_{-p}$$

RG transformation

$$S \longrightarrow S' \qquad (r \rightarrow r' = 2^2 r)$$



$$r^* : \quad r^* = 2^2 r^* \Rightarrow \underbrace{r^* = \overset{r_c}{\overset{\shortparallel}{0}}}_{\xi^* = \infty}, \ \underset{\xi^* = 0}{\infty}$$

$r = r_c$: "Gaussian fixed point"

$$S_c := \int_p \frac{1}{2} p^2 \tilde{\phi}_p \tilde{\phi}_{-p}$$

Critical exponent

$$\xi \sim \frac{1}{(K_c - K)^\nu} \qquad K? \quad r?$$

Change of the parameter $\quad r = f(K)$

assume it is analytic

$$r_c = f(K_c)$$

$$r_c - r = f(K_c) - f(K)$$

$$= \underbrace{f'(K_c)}_{\text{assume} \neq 0}(K_c - K) + \cdots$$

$$\Rightarrow \quad \xi \sim \frac{1}{(r-r_c)^{\nu}} \qquad \begin{array}{c} r > r_c \\ K < K_c \end{array} \Rightarrow (f'(K_c) < 0)$$

ν is the critical exponent

$$r_c = 0$$

$$\xi \sim \frac{1}{r^{\nu}}$$

RG

$$\xi' = \frac{1}{2}\xi, \quad r' = 2^2 r$$

$$\frac{1}{r'^{\nu}} = \frac{1}{2}\frac{1}{r} \Rightarrow \boxed{\nu = \frac{1}{2}}$$

Continuum limit

Original cutoff theory

$$r_0 \Rightarrow \xi_0$$

Introduce dimensionful lattice spacing a_0

$a_0 \to 0$ with appropreate dependence

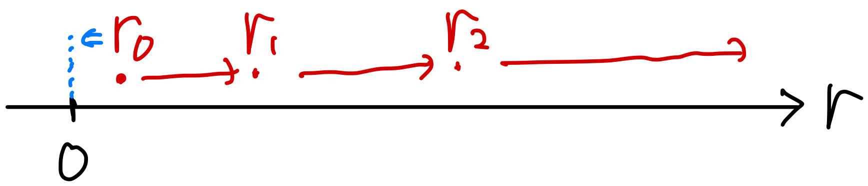
$r_0(a_0)$ "renormalization"

$a_0 \xi_0$ is finite (dimensionful "physical" correlation length)

$$a_0 \xi_0 \sim \frac{a_0}{r_0^{1/2}} \sim \frac{1}{m}$$

m : indep of a_0

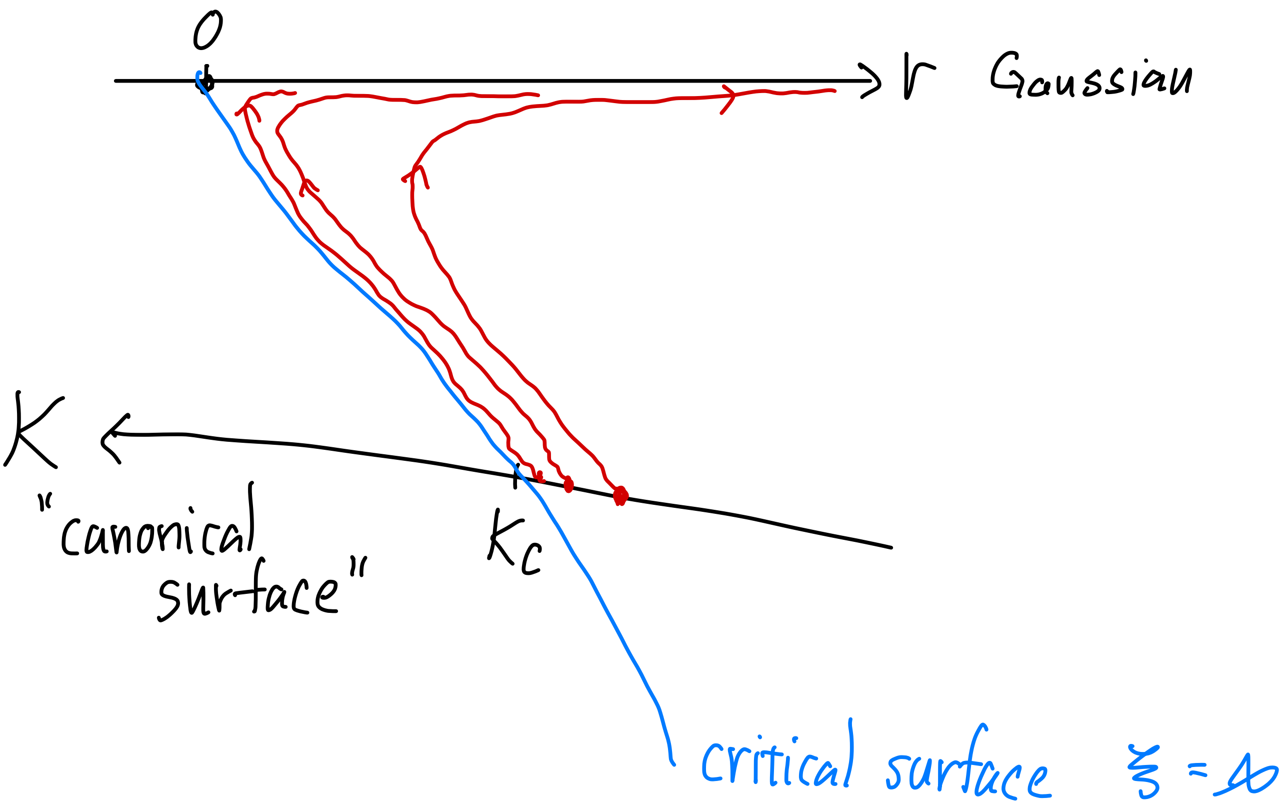
$$\Rightarrow \quad r_0 = m^2 a_0^2$$



※ $\nu = \frac{1}{2}$ is "universal"

If you start with a theory with $\mathbb{Z}_2$ symmetry. with a parameter K

You somehow know that starting $K = K_c \xrightarrow{RG}$ Gaussian fixed pt



$$\Rightarrow \quad \xi \sim \frac{1}{(K_c - K)^{\nu = \frac{1}{2}}}$$

$$K_0(a_0) = K_c - M^2 a_0^2 \quad \left(\begin{array}{c} M^2 : a_0 \text{ indep.} \\ \text{constant.} \end{array} \right)$$

$$a_0 \to 0 \quad \Rightarrow \quad \text{QFT}$$

※ Eg. $d > 4$ Ising model, ϕ^4-theory are known to be this type. "triviality"

QFT obtained here is a free field theory

$$\lim_{a_0 \to 0} a_0 \xi_0 = \frac{1}{m_{phys}} \leftarrow$$

mass of a particle, physical mass, ... pole mass, ...

This relation is not only for free theory.

☆ ϕ^4 model

$$S(\tilde{\phi}) = \underbrace{\int_p \frac{1}{2}(p^2+r)\tilde{\phi}_p \tilde{\phi}_{-p}}_{S_0} \qquad \int_p := \int_{|p|<1} \frac{d^d p}{(2\pi)^p}$$

$$+ \underbrace{\int_{p_1,\cdots,p_4} \frac{u}{4!} \tilde{\delta}(p_1+\cdots+p_4)\, \tilde{\phi}_{p_1} \cdots \tilde{\phi}_{p_4}}_{S_{Int}}$$

$$Z = \int D\tilde{\phi}\, e^{-S(\tilde{\phi})}$$

divide $\underbrace{|p|>\frac{1}{2}}_{\text{Integrate out}}$ and $|p|<\frac{1}{2}$

$$\tilde{\phi}_p = \rho_p + \chi_p \qquad \rho_p = \begin{cases} 0 & |p|>\frac{1}{2} \\ \tilde{\phi}_p & |p|<\frac{1}{2} \end{cases}$$

$$D\tilde{\phi} = D\rho\, D\chi \qquad \chi_p = \begin{cases} \tilde{\phi}_p & |p|<\frac{1}{2} \\ 0 & |p|>\frac{1}{2} \end{cases}$$

$$Z = \int D\rho\, \underbrace{D\chi\, e^{-S(\rho+\chi)}}_{\text{Integrate}}$$

$\Rightarrow$ ρ, χ do not decouple

$$= \int D\rho\, e^{-\tilde{S}(\rho)} \qquad e^{-\tilde{S}(\rho)} = \int D\chi\, e^{-S(\rho+\chi)}$$

$$\tilde{\phi}'_p := \zeta^{-1} \rho_{\frac{1}{2}p} \qquad S'(\tilde{\phi}') = \tilde{S}(\zeta\rho)$$

$(|p|<1)$ $\quad$ ↑ determined later

$$Z = \int D\tilde{\phi}\, e^{-S'(\tilde{\phi})}$$

RG transformation $\qquad S \longrightarrow S'$

□ Calculation

$$e^{-\tilde{S}(\rho)} = \int D\chi\, e^{-S(\rho+\chi)}$$

Exact calculation is difficult.

Try perturbation.

Free part

$$S_0(\rho+\chi) = \int_p \frac{1}{2}(p^2+r)(\rho_p+\chi_p)(\rho_{-p}+\chi_{-p})$$

$$= \int_p \frac{1}{2}(p^2+r)\rho_p\,\rho_{-p}$$

$$+ \int_p \frac{1}{2}(p^2+r)\chi_p\,\chi_{-p}$$

$$= S_0(\rho) + S_0(\chi)$$

Interaction part

$$S_{Int}(\varrho+\chi) = \int_{p_1,\cdots,p_4} \frac{u}{4!} \tilde{\delta}(p_1+\cdots+p_4)(\varrho_{p_1}+\chi_{p_1}) \cdots (\varrho_{p_4}+\chi_{p_4})$$

$$= \underbrace{S_{int,4}(\varrho)}_{\varrho^4} + \underbrace{S_{int,3}(\varrho,\chi)}_{\varrho^3\chi} + \underbrace{S_{int,2}(\varrho,\chi)}_{\varrho^2\chi^2} + \underbrace{S_{int,1}(\varrho,\chi)}_{\varrho\chi^3} + \underbrace{S_{int,0}(\chi)}_{\chi^4}$$

$$S_{int,4}(\varrho) = \int_{p_1,\cdots,p_4} \frac{u}{4!} \tilde{\delta}(p_1+\cdots+p_4)\varrho_{p_1}\cdots\varrho_{p_4}$$

$$S_{int,3}(\varrho,\chi) = \int_{p_1,\cdots,p_4} \frac{u}{3!} \tilde{\delta}(p_1+\cdots+p_4)\varrho_{p_1}\varrho_{p_2}\varrho_{p_3}\chi_{p_4}$$

$$S_{int,2}(\varrho,\chi) = \int_{p_1,\cdots,p_4} \frac{u}{4} \tilde{\delta}(p_1+\cdots+p_4)\varrho_{p_1}\varrho_{p_2}\chi_{p_3}\chi_{p_4}$$

$$S_{int,1}(\varrho,\chi) = \int_{p_1,\cdots,p_4} \frac{u}{3!} \tilde{\delta}(\cdots)\varrho_{p_1}\chi_{p_2}\chi_{p_3}\chi_{p_4}$$

$$S_{int,0}(\varrho,\chi) = \int_{p_1,\cdots,p_4} \frac{u}{4!} \tilde{\delta}(\cdots)\chi_{p_1}\cdots\chi_{p_4}$$

$$e^{-\tilde{S}(\rho)} = \int Dx \, e^{-S_0(\rho) - S_0(x) - S_{int}(\rho, x)}$$

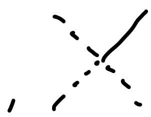
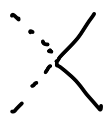
$$= e^{-S_0(\rho)} \int Dx \, e^{-S_0(x) - S_{int}(\rho, x)}$$

Propagator of x p $= \frac{1}{p^2+r}$

$\frac{1}{2} < |p| < 1$

External line of ρ

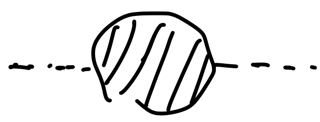
 p $= \int_p \rho_p$ $|p| < \frac{1}{2}$

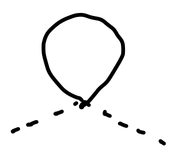
Vertex  $= -u$,  $= -u$,  $= -u$, ...

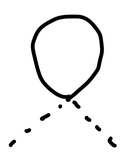
$$e^{-\tilde{S}(\rho)} = e^{-S_0(\rho)} \sum_{\text{all diagrams}} D_I$$

$$\Rightarrow \; -\tilde{S}(\rho) = -S_0(\rho) + \sum_{\substack{\text{all connected} \\ \text{diagrams}}} C_I$$

$$-\tilde{S}(\rho) = -\underset{\rho^2}{\underset{\wr}{\tilde{S}_2(\rho)}} - \underset{\rho^4}{\underset{\wr}{\tilde{S}_4(\rho)}} - \underset{\rho^6}{\underset{\wr}{\tilde{S}_6(\rho)}} - \cdots$$

$$-\tilde{S}_2(\rho) = -S_0(\rho) +$$ 

$$-\tilde{S}_2(\rho) = -S_0(\rho) +$$  $$+ \cdots$$

 $$= \int D\chi\, e^{-S_0(\chi)} \left(-S_{int,2}(\chi, \rho)\right)$$

$$= -\int_{p_1,\cdots,p_4} \frac{u}{4}\, \hat{\delta}(p_1+\cdots+p_4)\, \rho_{p_1} \rho_{p_2} \underset{\frac{1}{p_3^2+r}\hat{\delta}(p_3+p_4)}{\underset{\parallel}{\langle \chi_{p_3} \chi_{p_4} \rangle_0}}$$

$$= -\int_{p_1,p_2,p_3} \frac{u}{4} \rho_{p_1} \rho_{p_2} \tilde{\delta}(p_1+p_2) \times \frac{1}{p_3^2+r}$$

$p_1 =: p = -p_2$

$p_3 =: k$

$$= -\int_p \frac{1}{2} \rho_p \rho_{-p} \frac{u}{2} \int_{\substack{k \\ \frac{1}{2}<|k|<1}} \frac{1}{k^2+r}$$

define $$C(r) := \int_{\substack{k \\ \frac{1}{2}<|k|<1}} \frac{1}{k^2+r}$$

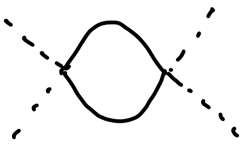
$$= -\int_p \frac{1}{2} \frac{u}{2} C(r) \rho_p \rho_{-p}$$

$$\tilde{S}_2(\rho) = \int_{\substack{p \\ |p|<\frac{1}{2}}} \frac{1}{2}\left(p^2+r+\frac{1}{2}u\,C(r)\right) \rho_p \rho_{-p} + O(u^2)$$

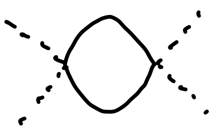
$$\tilde{\phi}'_p := 2^{-\frac{d+2}{2}} \varrho_{\frac{1}{2}p}$$

$$\tilde{S}_2(\varrho) = \int_p \frac{1}{2}(p^2 + r')\tilde{\phi}'_p \tilde{\phi}'_{-p} =: S'_0(\tilde{\phi}')$$

$$\boxed{r' = 4\left(r + \frac{u}{2}c(r)\right)}$$

$-\tilde{S}_4(\varrho) =$  $+$  $+ O(u^3)$

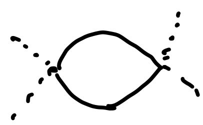
 $= -S_{int,4}(\varrho)$

 $= \left\langle \frac{1}{2}\left(-S_{int,2}(\varrho,\chi)\right)^2 \right\rangle_0$

$$= \frac{1}{2}\left(\frac{1}{4}\right)^2 u^2 \int_{p_1,\cdots,p_4} \int_{q_1,\cdots,q_4}$$

$$\times\ \hat{\delta}(p_1 + \cdots + p_4)\,\hat{\delta}(q_1 + \cdots + q_4)$$

$$\times\ \varrho_{p_1}\varrho_{p_2}\varrho_{q_1}\varrho_{q_2} \langle \chi_{p_3}\chi_{p_4}\chi_{q_3}\chi_{q_4} \rangle$$

2 ways of contraction to get 

$$= \frac{1}{2}\left(\frac{1}{4}\right)^2 \times 2\, u^2 \int_{p_1,\cdots,p_4} \int_{q_1,\cdots,q_4}$$

$$\times\ \tilde{\delta}(p_1+\cdots+p_4)\,\tilde{\delta}(q_1+\cdots+q_4)$$

$$\times\ \rho_{p_1}\rho_{p_2}\rho_{q_1}\rho_{q_2}\ \frac{1}{p_3^2+r}\,\tilde{\delta}(p_3+q_3)\,\frac{1}{p_4^2+r}\,\tilde{\delta}(p_4+q_4)$$

Integrate out q_3, q_4, p_4 using $\tilde{\delta}$

$q_3 = -p_3$, $q_4 = -p_4 = p_1+p_2+p_3$

$$= \frac{1}{4!}\int_{p_1,p_2,q_1,q_2} \tilde{\delta}(p_1+p_2+q_1+q_2)\, V_{4,1}(p_1,p_2,q_1,q_2)$$

$$\times\ \rho_{p_1}\rho_{p_2}\rho_{q_1}\rho_{q_2}$$

$$V_{4,1} = 3\times\frac{1}{2}u^2 \int_{\substack{k \\ \frac{1}{2}<|k|<1 \\ \frac{1}{2}<|k+p_1+p_2|<1}} \frac{1}{k^2+r}\,\frac{1}{(k+p_1+p_2)^2+r}$$

$$= 3\times\frac{1}{2}u^2 \int_{\substack{k \\ \frac{1}{2}<|k|<1}} \frac{1}{(k^2+r)^2} \quad + \underline{O(p^2)}$$

Ignore

Justified in some cases

$$C(r) := \int_{\substack{k \\ \frac{1}{2}<|k|<1}} \frac{1}{k^2+r} \Rightarrow \int_k \frac{1}{(k^2+r)^2} = -C'(r)\left(= -\frac{d}{dr}C(r)\right)$$

$$V_{4,1} = \frac{3}{2} u^2 \left(-C'(r) \right) + O(p^2)$$

$$\tilde{S}_4(\varphi) = \frac{1}{4!} \int_{\substack{p_1, \cdots, p_4 \\ |p_i| < \frac{1}{2}}} \tilde{\delta}(p_1 + \cdots + p_4) \left(u + \frac{3}{2} u^2 C'(r) + O(p^2) \right) \times \varphi_{p_1} \cdots \varphi_{p_4} \quad + O(u^3)$$

(scaling)

$$= \frac{1}{4!} \int_{\substack{p_1, \cdots, p_4 \\ |p_i| < 1}} \tilde{\delta}(p_1 + \cdots + p_4) \left(u' + O(p^2) \right) \tilde{\phi}'_{p_1} \cdots \tilde{\phi}'_{p_4}$$

$$= S_4'(\tilde{\phi}')$$

$$\boxed{u' = 2^{4-d} \left(u + \frac{3}{2} u^2 C'(r) \right)}$$

RG transformation at 1-loop

$$\boxed{\begin{aligned} r' &= 4 \left(r + \frac{u}{2} C(r) \right) \\ u' &= 2^{4-d} \left(u + \frac{3}{2} u^2 C'(r) \right) \end{aligned}}$$

※ In general, infinitely many terms appears.
Here, we focus on r, u.
This is justified when $4-d := \varepsilon \ll 1$
We can also gain a qualitative understanding.

※ Due to the symmetry $\tilde{\phi} \to -\tilde{\phi}$, only even power appears.

Fixed points

$r=0$, $u=0$ is a fixed point.

"Gaussian fixed point"

Other fixed points?

$d>4$ small u

$$u' = 2^{4-d} u \Rightarrow u \text{ decreasing}$$

$\Rightarrow$ go to Gaussian fixed point.

$d=4$

$$u' = u + \frac{3}{2} u^2 \underbrace{C'(r)}_{<0} \Rightarrow \text{decreasing}$$

$d<4$

$$u' = 2^{4-d} u \Rightarrow \text{increasing.}$$

possible another fixed point

Assume r_*, u_* small $\Rightarrow$ $C(r) = A - Br$

$A, B > 0$ constant

$$r_* = 4\left(r_* + \frac{u_*}{2}(A - Br_*)\right)$$

$$u_* = 2^{4-d}\left(u_* - \frac{3}{2}u_*^2 B\right) \qquad 4-d := \varepsilon$$

$(u_* \neq 0)$

$$1 = 2^{\varepsilon}\left(1 - \frac{3}{2}u_* B\right)$$

$$2^{-\varepsilon} = 1 - \frac{3}{2}u_* B$$

ε small $\Rightarrow$ u_* small

perturbation is valid

$$u_* = \frac{2}{3B}\left(1 - 2^{-\varepsilon}\right)$$

"ε-expansion"

$$u_* = \frac{2}{3B}\varepsilon \log 2 + O(\varepsilon^2)$$

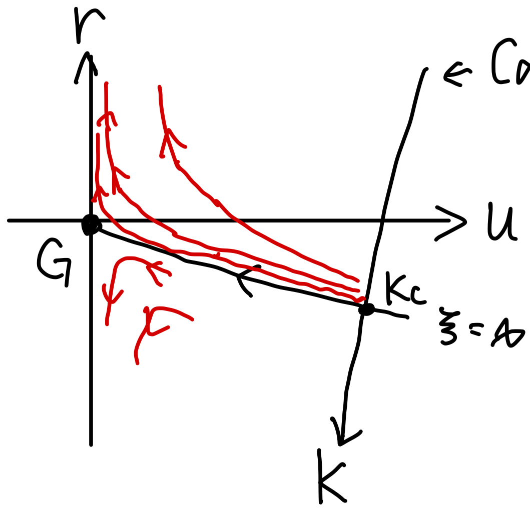
$$r_* = -\frac{1}{6}Au^* + O(\varepsilon^2)$$

$$= -\frac{A}{9B}\varepsilon \log 2 + O(\varepsilon^2)$$

"Wilson-Fisher fixed point"

Our class of theory : 1 scalar field, $\mathbb{Z}_2$ symmetry
(Including Ising model)

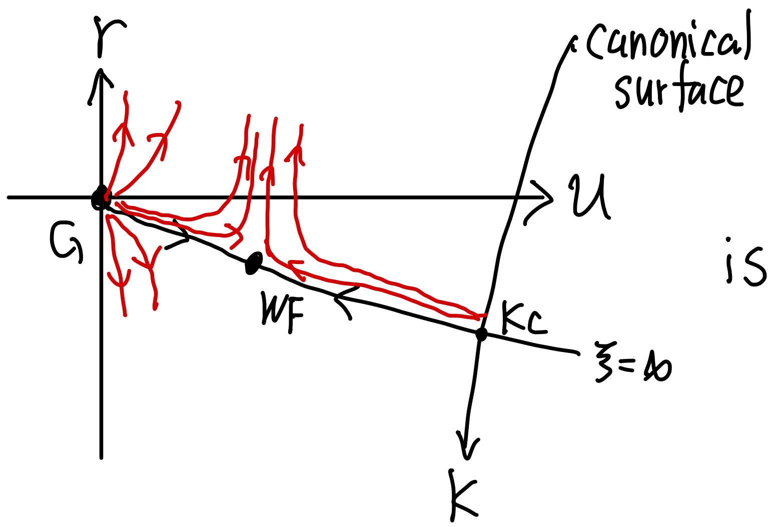
$d \geq 4$



Original cutoff action is given
(Ising model)

critical phenomena
(long range physics
around $K \sim K_c$)
is governed by the Gaussian
fixed point.

$d < 4$
(at least $d = 4 - \varepsilon$, $\varepsilon \ll 1$)



critical phenomena
is governed by the WF
fixed point

$\Rightarrow$ Need to know the flow
around WF.

Linearized flow equation

$d = 4 - \varepsilon, \quad (\varepsilon \ll 1)$

$$r' = 4\left(r + \frac{u}{2}(A - Br)\right)$$

$$u' = 2^{\varepsilon}\left(u - \frac{3}{2}u^2 B\right)$$

$\delta r = r - r_*, \quad \delta u = u - u_*$ $\qquad \delta r, \delta u \ll 1$

$\Rightarrow$ approximated by the linearized equation.

$\Downarrow$

$$\delta r' = 4\left(\delta r + \frac{1}{2}u_*(-B)\delta r + \frac{1}{2}\delta u(A - Br_*)\right)$$

$$= (4 - 2Bu_*)\delta r + 2\underbrace{(A - Br_*)}_{A'}\delta u$$

$$= \left(4 - \frac{4}{3}\varepsilon \log 2\right)\delta r + 2A'\delta u \qquad Bu_* = \frac{2}{3}\varepsilon \log 2$$

$$\delta u' = 2^{\varepsilon}(1 - 3u_* B)\delta u$$

$$= (1 + \varepsilon \log 2)(1 - 2\varepsilon \log 2)\delta u$$

$$= (1 - \varepsilon \log 2)\delta u$$

$\Downarrow$

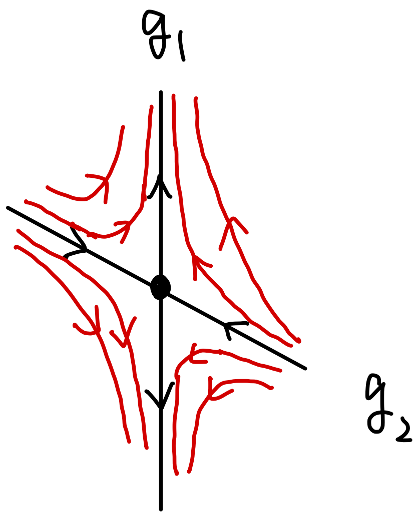
$$\begin{pmatrix} \delta r' \\ \delta u' \end{pmatrix} = M \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}, \quad M = \begin{pmatrix} 4 - \frac{4}{3}\varepsilon \log 2 & 2A' \\ 0 & 1 - \varepsilon \log 2 \end{pmatrix}$$

$\Rightarrow$ It is useful to diagonalize M

Eigenvalue $\lambda_1 = 4 - \frac{4}{3}\varepsilon \log 2$ $\lambda_2 = 1 - \varepsilon \log 2$

(>1) (<1)

Eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} \frac{-2A'}{\lambda_1 - \lambda_2} \\ 1 \end{pmatrix}$



$$\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = g_1 v_1 + g_2 v_2$$

$g_1' = \lambda_1 g_1$ $(\lambda_1 > 1)$ growing

"relevant" coupling
deformation

$g_2' = \lambda_2 g_2$ $(\lambda_2 < 1)$ decreasing

"irrelevant" coupling
deformation

※ If $\lambda = 1$, (eg. $d=4$ coupling u),

"maginal" coupling or deformation

look at higher order

decreasing "marginally irrelevant" this eg.

growing "marginally relevant"

do not flow "exactly marginal"

※ Such classification depends on the fixed point.
For generic point, we cannot say a coupling is relevant or irrelevant.

□ Critical exponent

$$g_1 \quad : \quad g_1' = \lambda_1 g_1$$

$$\lambda_1 = 4 - \frac{4}{3}\varepsilon \log 2$$

$K(g_1, g_2)$ analytic

$$K - K_c = \frac{\partial K}{\partial g_1}(0,0)\, g_1 + \cdots$$

$$\xi \sim \frac{1}{(K_c - K)^\nu} \sim \frac{1}{g_1^\nu}$$

RG transf.

$$\xi' = \frac{1}{2}\xi \quad , \quad g_1' = \lambda_1 g_1$$

$$\Rightarrow \frac{1}{2 g_1^\nu} = \frac{1}{(\lambda_1 g_1)^\nu}$$

$$\Rightarrow \frac{1}{2} = \lambda_1^{-\nu}$$

$$-\log 2 = -\nu \log \lambda_1$$

$$\nu = \frac{\log 2}{\log \lambda_1}$$

$$\log \lambda_1 = \log\left(4\left(1-\frac{1}{3}\varepsilon \log 2\right)\right)$$

$$= 2\log 2 + \log\left(1-\frac{1}{3}\varepsilon \log 2\right)$$

$$= (\log 2)\left(2-\frac{1}{3}\varepsilon\right)$$

$$\nu = \frac{\log 2}{\log \lambda_1} = \frac{1}{2-\frac{1}{3}\varepsilon} = \frac{1}{2} + \frac{1}{12}\varepsilon$$

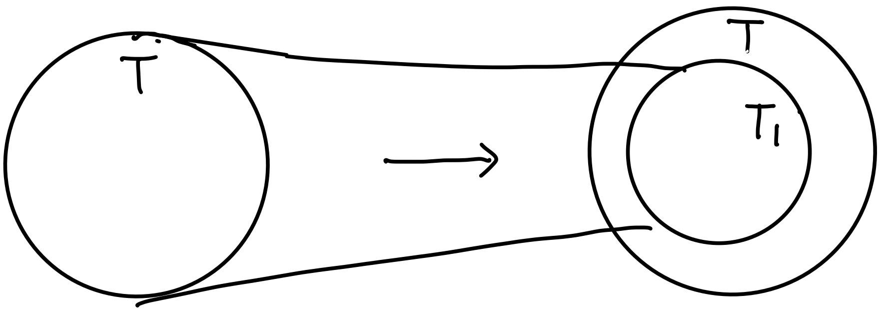
☆ Structure of the space of theories

T : set of cutoff theories (with certain symmetry)
Local interaction
(exponential decay is allowed)

<u>Eg.</u> $S \sim \int d^dx\, d^dy\, e^{-\ell|x-y|} \phi(x)\phi(y)$

∞ possible terms $\quad \ell \sim O(1)$

$RG : T \to T$



$\mathrm{Im}\, RG =: T_1 \subset T \Rightarrow \mathrm{Im}(RG)^n =: T_n$

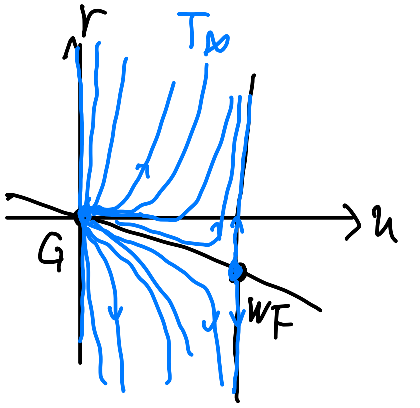
$T_1 \supset T_2 \supset \cdots$

$T_\infty := \bigcap_n T_n$

$S \in T_\infty$

$\Leftrightarrow$ arbitorary small cutoff,
$\exists S_0$ whose long range physics
is the same as S

<u>Eg.</u> ϕ^4, $d = 4 - \varepsilon$

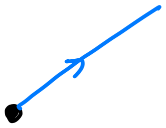


$\sim$ continuous QFT

$T_\infty \ni$ {fixed points}

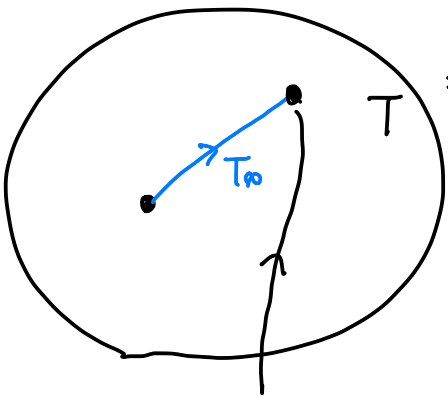
$S_* = S_*$

$\xi = 0$ or ∞



a finite dimensions of relevant directions $\in T_\infty$

others are irrelevant (?)

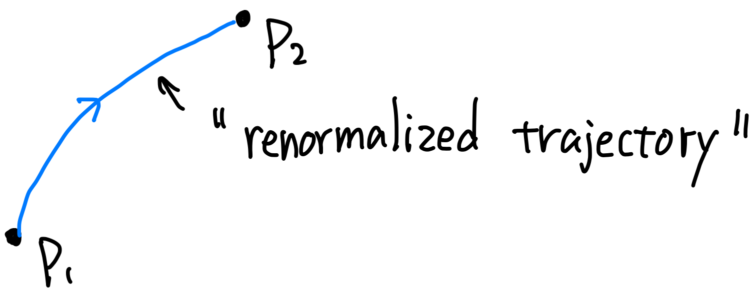


For $\forall S \in T_\infty$

$\exists$ fixed pt P_1 in the upstream

P_2 in the downstream (generic)

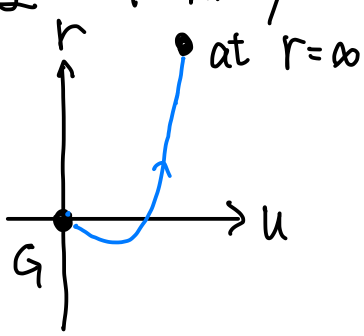
(often "empty" $\xi = 0$)

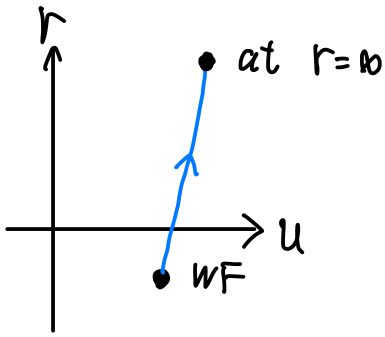


<u>claim</u>

$$\boxed{\text{continuous QFT (with scale)} \Longleftrightarrow \text{renormalized trajectory}}$$

<u>Eg.</u> ϕ^4 theory in $d = 4 - \varepsilon$





Continuum limit

T, T_∞, action includes ∞ of terms, in general.

Can we get continuous QFT from simple action?

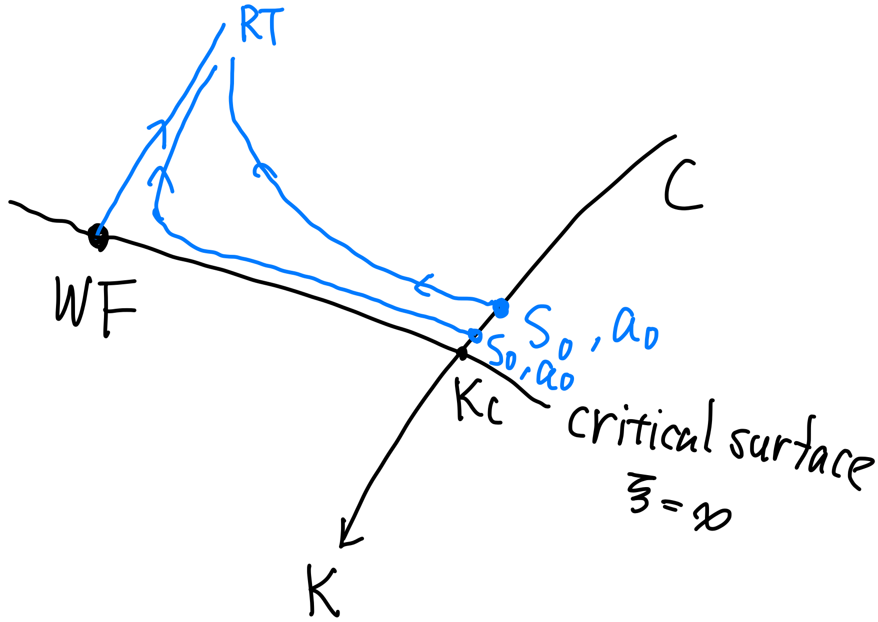
"Canonical surface" $C \subset T$

simple cut off action is given.

Eg. Ising model,

ϕ^4 theory, ...

Eg. Ising, $d = 4 - \varepsilon$ dimensions (d=3 is the same qualitatively)



a_0: dimensionful lattice spacing

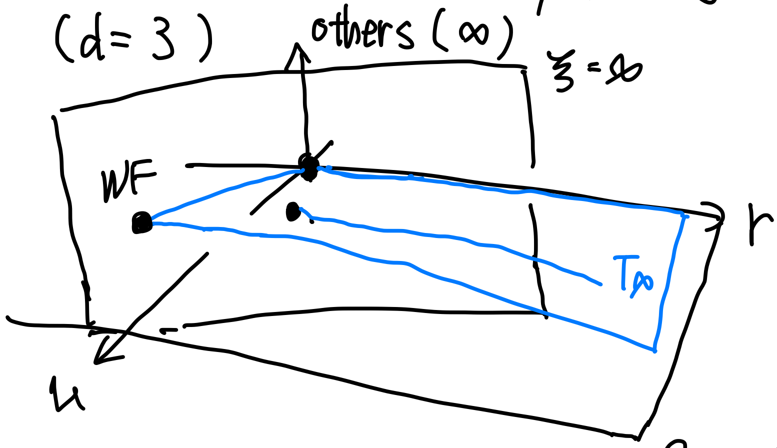
Renormalization

$S_0 \in C$, a_0 cutoff theory
(K)

correlation length $\xi_0(K)$

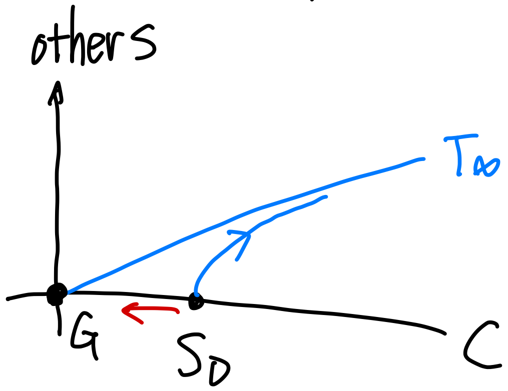
$K \to K_c$, $a_0 \to 0$ with $\xi_0(K) a_0$ fixed

Eg. $d = 4 - \varepsilon$, ϕ^4 theory, using Gaussian fixed point
($d = 3$)



C : (others = 0)

$S_0 \in C$ (r_0, u_0)



2 relevant couplings

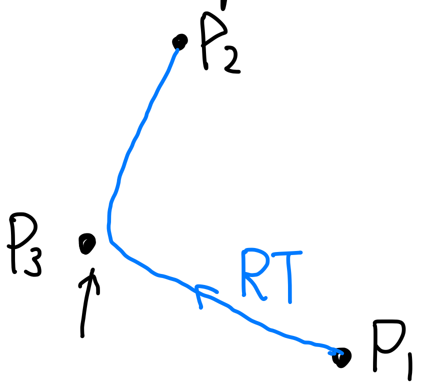
$\Rightarrow$ In addition to $\xi_0 a_0$, there is another parameter. $\Leftarrow$ tune r_0, u_0

☆ Effective theory

QFT defined in the limit is called "UV complete QFT"

What we call "QFT" is not limited to UV complete QFT

A UV complete QFT (or something beyond QFT)



Theory stay here long (not ∞) time.

$\Rightarrow$ Low energy physics (below an energy scale Λ)
(Larger length scale than $\ell = \frac{1}{\Lambda}$)

is approximately governed by fixed point P_3,

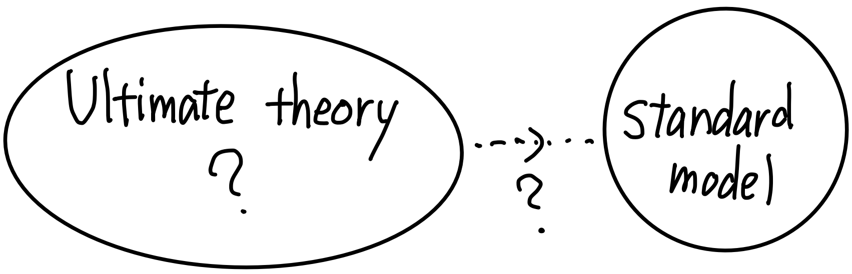
without knowing much about P_1 (or something beyond QFT)

Around P_3

relevant couplings
(marginal)) finite number

irrelevant couplings $\Rightarrow$ small in low energy
but not zero

Eg: Real world (?)



$\rightarrow$
- mass (relevant)
- QED (marginal)
- Four-Fermi (irrelevant)
- QCD (marginally relevant)

7. Symmetry

☆ Current and charge in QFT

Euclidean action

$$S_E(\phi) = \int d^dx\, \mathcal{L}_E(x)$$

Consider infinitesimal transf.

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \delta\phi(x)$$

$$\delta\phi(x) = \varepsilon\, \Delta\phi(x)\,, \quad \varepsilon: \text{constant, infinitesimal parameter}$$

If $\delta\mathcal{L}_E = \varepsilon\, \partial_\mu K^\mu$. $\exists K^\mu$

$\Rightarrow$ This transf. is called a "symmetry."

(We do NOT assume $S_E(\phi)$ is invariant)

$\Downarrow$

current, charge, Ward-Takahashi identity (WT)

Eg. Complex scalar $\phi(x)$

$$\mathcal{L}_E = \partial_\mu \phi^* \partial^\mu \phi + V(|\phi|^2)$$

$$\delta\phi(x) = i\varepsilon\phi(x) \quad \Rightarrow \delta\phi^*(x) = -i\varepsilon\phi^*(x)$$

$$(\Delta\phi(x) = i\phi(x)) \qquad (\Delta\phi^*(x) = -i\phi^*(x))$$

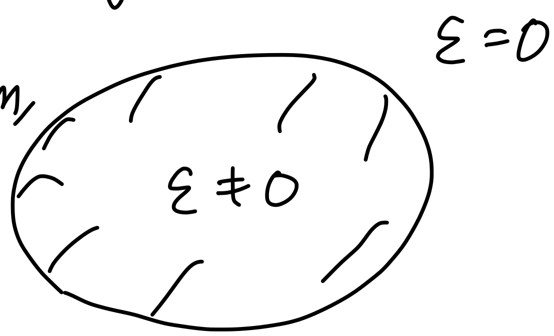
$$\Rightarrow \delta\mathcal{L}_E = 0$$

Current

Consider a position dependent parameter $\varepsilon(x)$

Choose $\varepsilon(x) = 0$ outside of a finite region

$\Rightarrow$ You can ignore a surface term, when partial integral.



$$\tilde{\delta}\phi(x) = \varepsilon(x)\Delta\phi(x)$$

$$\tilde{\delta} S_E(\phi) = \int d^dx\, \tilde{\delta}\mathcal{L}_E$$

"Symmetry"

$$\tilde{\delta}\mathcal{L}_E = \varepsilon(x)\partial_\mu K^\mu + (\partial_\mu \varepsilon)\,\mathrm{I} + \partial_\mu \mathrm{II}$$

$$= -\partial_\mu\varepsilon(x) J^\mu(x) \quad + \partial_\mu \mathrm{II}$$

$\exists J^\mu(x)$: "(Noether) current"

$$\tilde{\delta} S_E = -\int d^d x \left(\partial_\mu \varepsilon(x) \right) J^\mu(x)$$

$$\boxed{\tilde{\delta} S_E = \int d^d x \; \varepsilon(x) \, \partial_\mu J^\mu(x)} \qquad \cdots (*)$$

Eg. complex scalar

$$S_E = \int d^d x \left(\partial_\mu \phi^* \partial_\mu \phi + V(|\phi|^2) \right)$$

$$\tilde{\delta} |\phi|^2 = \tilde{\delta}(\phi^* \phi) = \tilde{\delta}\phi^* \phi + \phi^* \tilde{\delta} \phi$$

$$= -i \varepsilon(x) \phi^* \phi + \phi^* i \varepsilon(x) \phi = 0$$

$$\tilde{\delta} V(|\phi|^2) = 0$$

$$\tilde{\delta} \mathcal{L}_E = \partial_\mu (\tilde{\delta} \phi^*) \partial_\mu \phi + \partial_\mu \phi^* \partial_\mu \tilde{\delta} \phi$$

$$= -i \, \partial_\mu (\varepsilon \phi^*) \partial_\mu \phi + i \, \partial_\mu \phi^* \partial_\mu (\varepsilon \phi)$$

$$= (\partial_\mu \varepsilon)(-i \phi^* \partial_\mu \phi + i \phi \partial_\mu \phi^*)$$

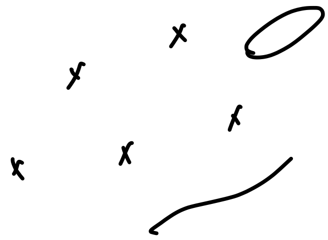
$$J^\mu = i \phi^* \partial^\mu \phi - i \phi \partial^\mu \phi^*$$

□ WT id

Expectation value

$$\langle X \rangle := \frac{1}{Z} \int D\phi \, X \, e^{-S_E(\phi)}$$

X: arbitrary operator



$$= \frac{1}{Z} \int D\phi' \, X' \, e^{-S_E(\phi')}$$

(change the character of the integration variables)

$$\left(\phi' = \phi + \tilde{\delta}\phi, \quad \text{assume } D\phi' = D\phi \right)$$

$$= \frac{1}{Z} \int D\phi \, (X + \tilde{\delta} X) \, e^{-S_E(\phi) - \tilde{\delta} S_E(\phi)}$$

$$= \langle X \rangle + \langle \tilde{\delta} X \rangle - \langle \tilde{\delta} S_E(\phi) \, X \rangle$$

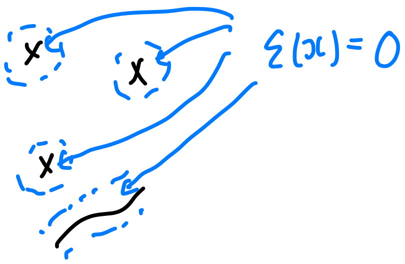
$$\Rightarrow \quad \langle \tilde{\delta} S_E(\phi) \, X \rangle = \langle \tilde{\delta} X \rangle$$

(*)

$$\Rightarrow \boxed{\int d^d x \; \varepsilon(x) \langle \partial_\mu J^\mu(x) X \rangle = \langle \tilde{\delta} X \rangle} \quad \cdots (\#)$$

Consider various choice of $\varepsilon(x)$

(1) $\varepsilon(x) = 0$ in neighborhood of the locus of X



$$\Rightarrow \tilde{\delta} X = 0$$

$$\Rightarrow \int d^d x \; \varepsilon(x) \langle \partial_\mu J^\mu(x) X \rangle = 0$$

$\varepsilon(x)$ is arbitrary besides the conditions stated.

$$\Rightarrow \langle \partial_\mu J^\mu(x) X \rangle = 0$$

if x does not coincide with a locus of X

This is often abbreviated as

$$\boxed{\partial_\mu J^\mu(x) = 0}$$

(2) If

$$X = \mathcal{O}_1(x_1)\, Y \quad \leftarrow \text{arbitrary}$$

$\mathcal{O}_1(x_1)$: a local operator

$\varepsilon(x)$: constant in nbhd of x_1

0 in nbhd of the locus of Y

for constant ε

$$\delta \mathcal{O}_1(x_1) = \varepsilon \Delta \mathcal{O}_1(x_1)$$

$$\Rightarrow \tilde{\delta} \mathcal{O}_1(x_1) = \varepsilon(x_1)\, \Delta \mathcal{O}_1(x_1)$$

$$\Rightarrow \tilde{\delta} X = \tilde{\delta} \mathcal{O}_1(x_1) Y = \varepsilon(x_1) \Delta \mathcal{O}_1(x_1)$$

$$(\#) \Rightarrow \int d^d x\, \varepsilon(x) \langle \partial_\mu J^\mu(x)\, \mathcal{O}_1(x_1)\, Y \rangle$$

$$= \varepsilon(x_1) \langle \Delta \mathcal{O}_1(x_1)\, Y \rangle$$

$$= \int d^d x\, \varepsilon(x)\, \delta^d(x - x_1) \langle \Delta \mathcal{O}_1(x_1)\, Y \rangle$$

$$\Rightarrow \begin{cases} \langle \partial_\mu J^\mu(x)\, \mathcal{O}_1(x_1)\, Y \rangle = \delta^d(x - x_1) \langle \Delta \mathcal{O}_1(x_1)\, Y \rangle \\ \text{if } x \text{ does not coincide with loc. } Y \end{cases}$$

abbr.

$$\boxed{\partial_\mu J^\mu(x)\, \mathcal{O}_1(x_1) = \delta^d(x - x_1)\, \Delta \mathcal{O}_1(x_1)} \quad \cdots\ 4$$

"WT id"

※ WT id holds even if the symmetry is sponteneously broken.

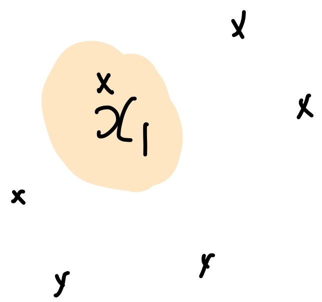
Charge

Σ : codimension 1, oriented surface in spacetime

$$Q(\Sigma) := -i \int_\Sigma d\Sigma_\mu \, J^\mu(x)$$

(surface integral you learned in electromagnetism.)

D : a region including x_1, does not include loc. of Y.



$\Sigma = \partial D$

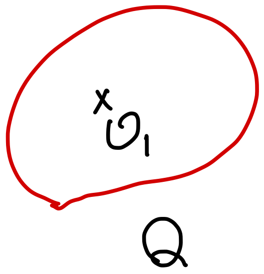
$\int_D d^dx \; (4)$

rhs $= \Delta \mathcal{O}_1(x_1)$

lhs $= \int_D d^dx \; \partial_\mu J^\mu(x) \, \mathcal{O}_1(x_1)$

$$\underset{\substack{\uparrow \\ \text{Gauss's thm.}}}{=} \int_\Sigma d\Sigma_\mu \, J^\mu(x) \, \mathcal{O}_1(x_1) = i \, Q(\Sigma) \, \mathcal{O}_1(x_1)$$

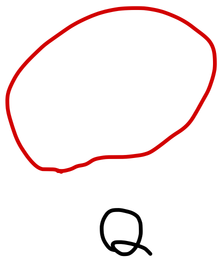
$$\boxed{iQ(\Sigma)\,\mathcal{O}_1(x_1) = \Delta\mathcal{O}_1(x_1)}$$

i 

$= \overset{\times}{\Delta\mathcal{O}_1}$

WT id

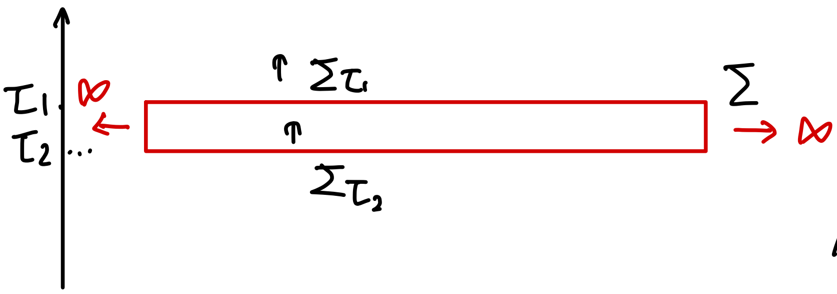
If $\mathcal{O}_1 = 1$ (or integrate $\partial_\mu J^\mu = 0$ in D)



$= 0$

☐ Back to operator formalism

τ : Euclidean time



No other op. is inserted between τ_1 and τ_2

$$\Sigma = \Sigma_{\tau_1} \cup (-\Sigma_{\tau_2})$$

$$\underset{\substack{\shortparallel \\ Q(\tau_1)}}{Q(\Sigma_{\tau_1})} = -i\int_{\Sigma_{\tau_1}} d\Sigma_\mu J^\mu = -i\int d^{d-1}\vec{x}\; J^\tau(\tau_1, \vec{x})$$

Including integral over infinite region ⇒ may be subtle.
(cf SSB)

Assume $Q(\tau_1)$ is well-defined.

$$0 = Q(\Sigma) = Q(\tau_1) - Q(\tau_2)$$

$\Downarrow$ operator formalism

arbitrary

$$0 = \langle 0 | \cdots \left(\hat{Q}(\tau_1) - \hat{Q}(\tau_2) \right) \cdots | 0 \rangle$$

Euclidean time ordering

$$\hat{Q}(\tau_1) - \hat{Q}(\tau_2) = 0 \quad \text{as an operator}$$

$$\hat{Q}(\tau_1) = \hat{Q}(\tau_2) \qquad (\tau \text{ independent})$$

$$=: \hat{Q} \qquad ((\text{Noether}) \text{ charge})$$

$$\boxed{\hat{Q} \text{ is conserved}}$$

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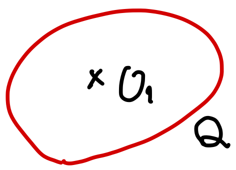
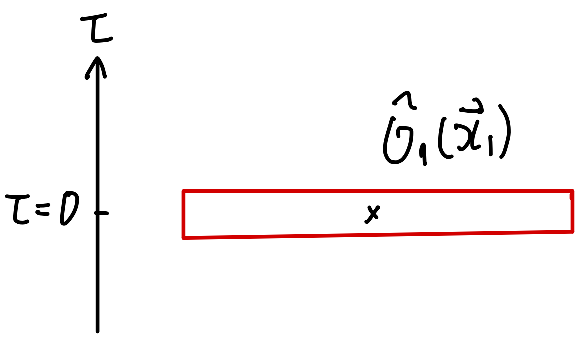
i 

$=$ $\times\ \Delta O_1$



$$Q(\Sigma) = \hat{Q}(\tau_1) - \hat{Q}(\tau_2)$$

$$\tau_1 > 0\,, \quad \tau_2 < 0$$

Operator formalism

$$\langle 0| \cdots \left(i\underbrace{\hat{Q}(\tau_1)}_{\hat{Q}}\, \hat{\mathcal{O}}_1(\vec{x}_1) - \hat{\mathcal{O}}_1(\vec{x}_1)\, i\underbrace{\hat{Q}(\tau_2)}_{\hat{Q}} \right) \cdots |0\rangle$$

$$= \langle 0| \cdots \Delta \hat{\mathcal{O}}_1(\vec{x}_1) \cdots |0\rangle$$

$$\Rightarrow \quad i\left[\hat{Q}, \hat{\mathcal{O}}_1(\vec{x})\right] = \Delta \hat{\mathcal{O}}_1(\vec{x})$$

$\hat{Q}$ is the generator of the symmetry!